Advanced
Competitive Programming

國立成功大學ACM-ICPC程式競賽培訓隊
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Made by 培訓團隊群 CCNS.

Outline

- 術語複習
  - Graph
  - Tree
- 最小生成樹
- A* 搜尋法則
- 單源最短路徑
- 全點對最短路徑

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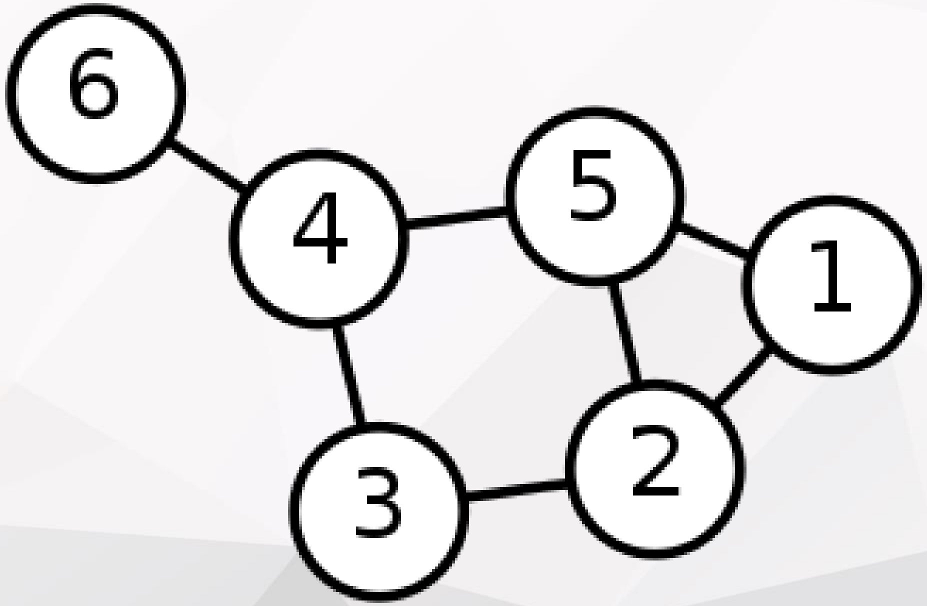
Graph

- 圖 (Graph)，是一個由邊 (Edge) 集合與點 (Vertex) 集合所組成的資料結構。



Graph

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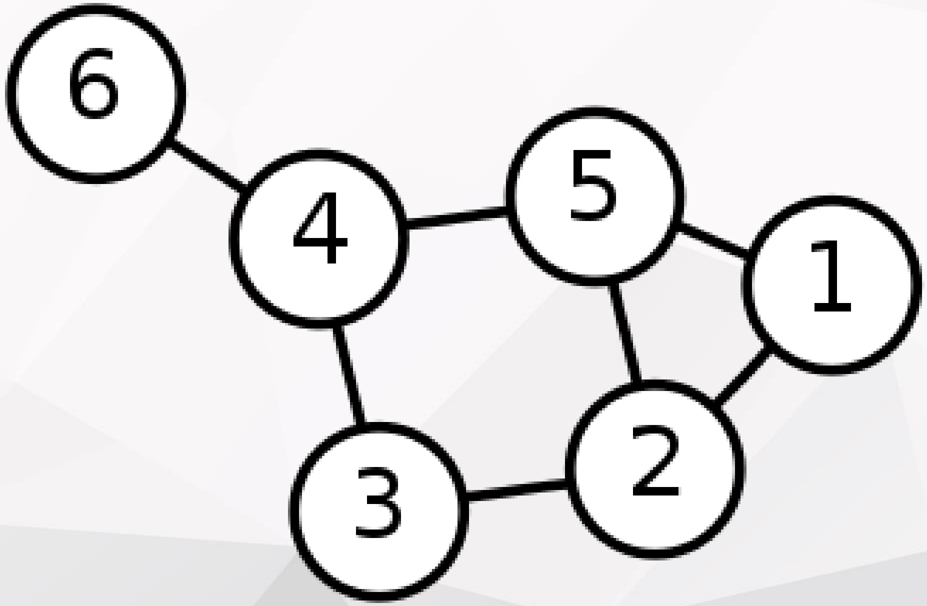
- G = (E, V) 為圖
- E 為邊集合
- V 為點集合



Graph

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- 有向圖 (directed graph)： 邊帶有方向性
  - u -> v，表示 u 能走到 v，但 v 不保證走到 u
- 無向圖 (undirected graph)： 每條邊都是雙向的
  - u <-> v，表示 v 能到 u，u 能到 v

Graph

- 道路 (walk)： 點邊相間的序列

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Graph

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Graph

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  e.g. $v_0e_1v_1e_2v_2 \ldots e_nv_n$
- 路徑 (path)： 點不重複的道路
- 環 (cycle)： 把路徑的**起**點與**終**點連接起來
- 走訪/遍歷 (traversal/search)： 走完全部的點或邊

該怎麼表示一張圖呢

Graph 鄰接矩陣

- 用二維陣列表達點與點有無**邊關係**
  - E[u][v] = 1 表示 u 與 v 間有邊
  - E[u][v] = 0 表示 u 與 v 間沒邊

Graph 鄰接矩陣

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- 用特殊的值來表示**無法到達**的情況
  - 例如 INT_MAX 或是 -1 或是 INF

Graph 鄰接表

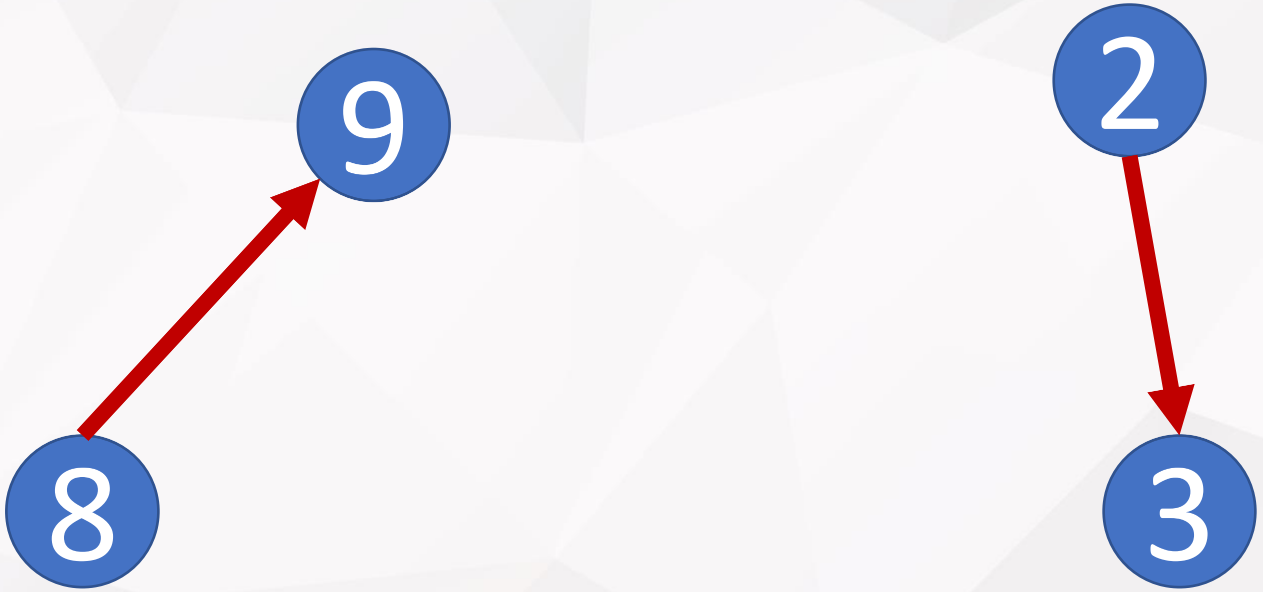
```
vector<int> E[maxv];

to = E[from][0];
```

Graph 鄰接表

```
E[2][0] = 3;
```

Graph 鄰接表

```
E[2][0] = 3;
E[8][0] = 9;
```

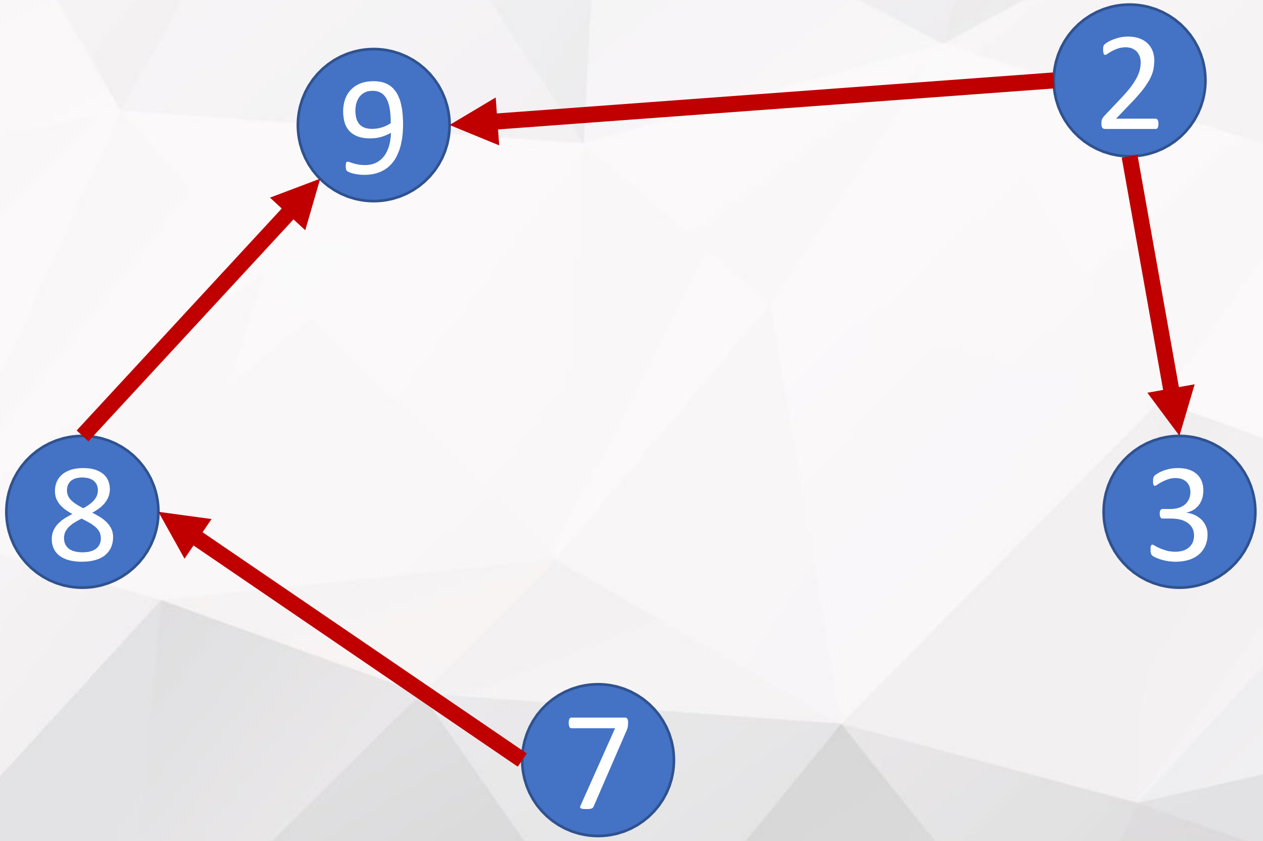


Graph 鄰接表

```
E[2][0] = 3;
E[8][0] = 9;
E[2][1] = 9;
```

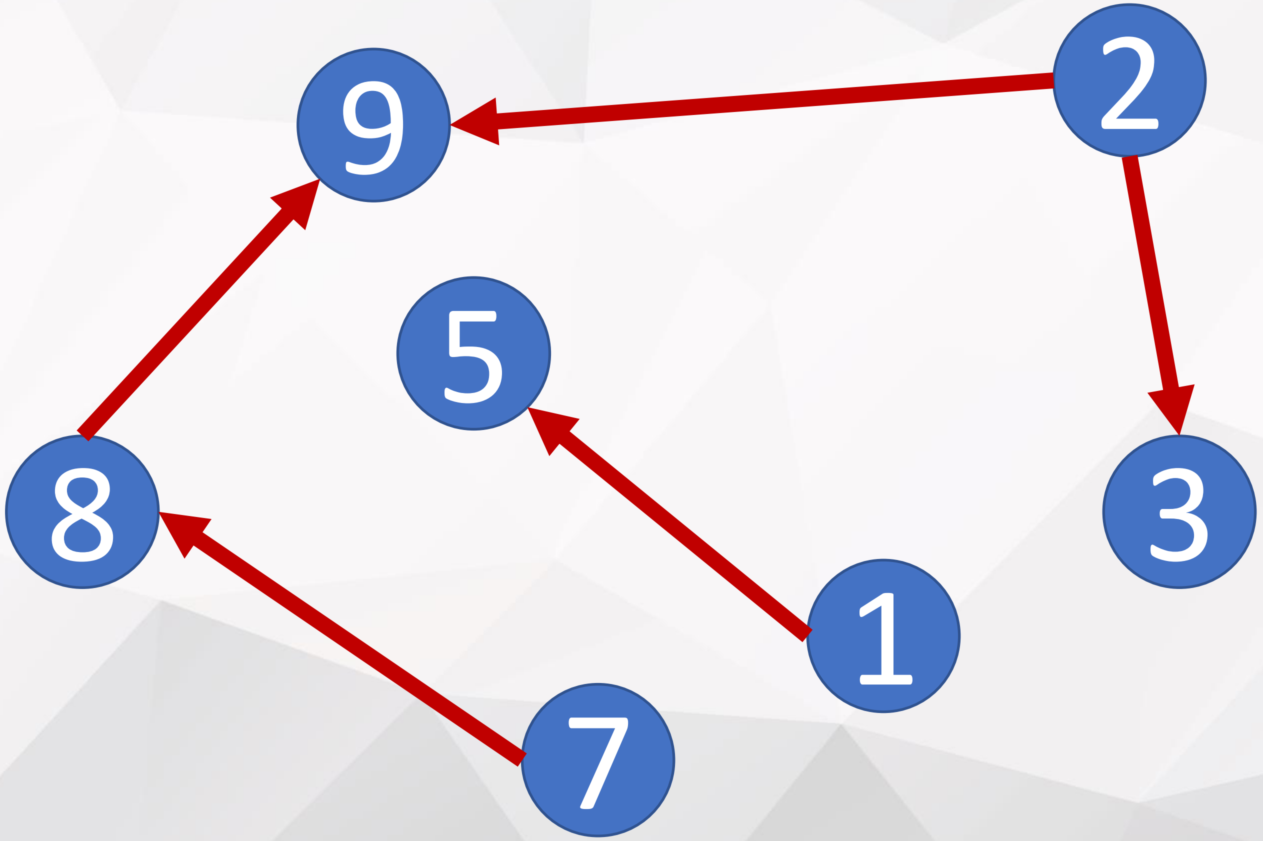
Graph 鄰接表

```
E[2][0] = 3;
E[8][0] = 9;
E[2][1] = 9;
E[7][0] = 8;
```



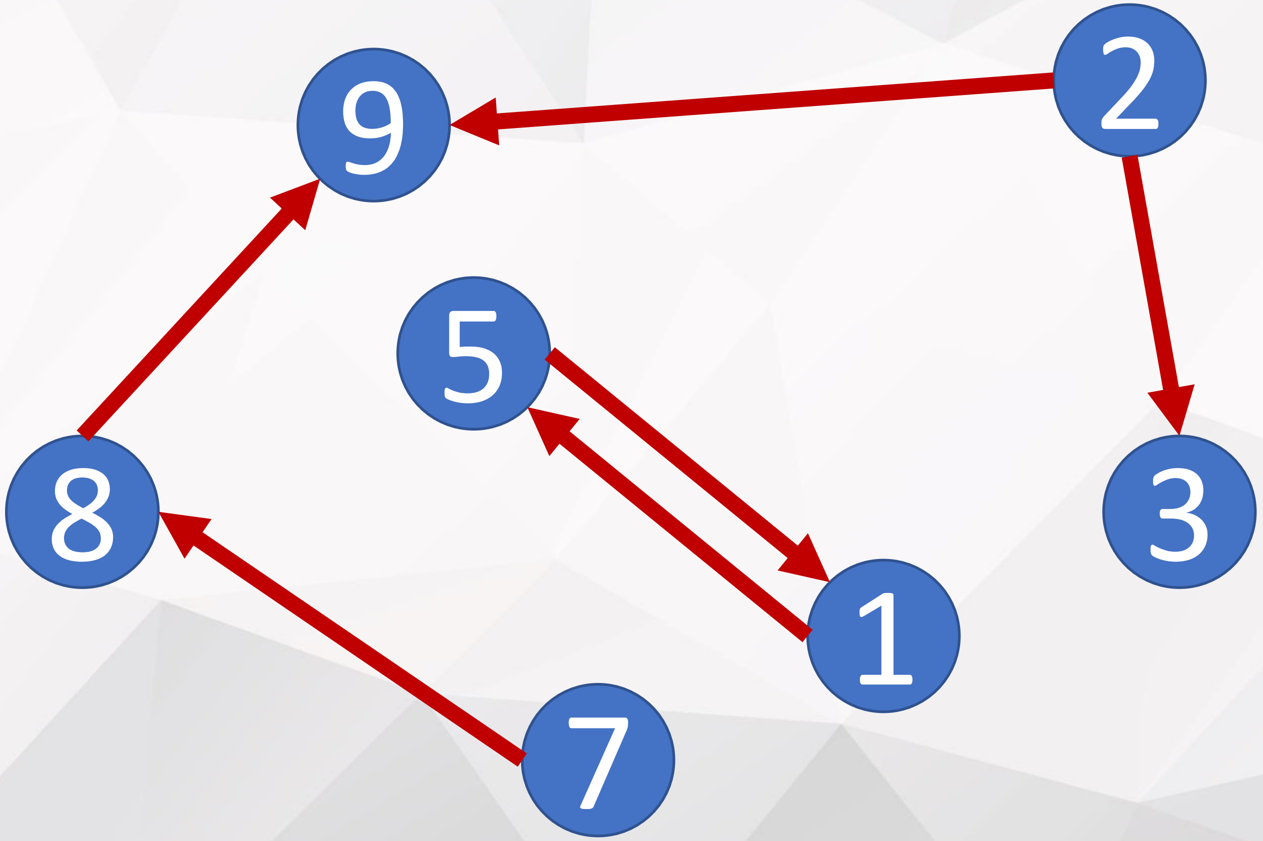
Graph 鄰接表

```
E[2][0] = 3;
E[8][0] = 9;
E[2][1] = 9;
E[7][0] = 8;

E[1][0] = 5;
```



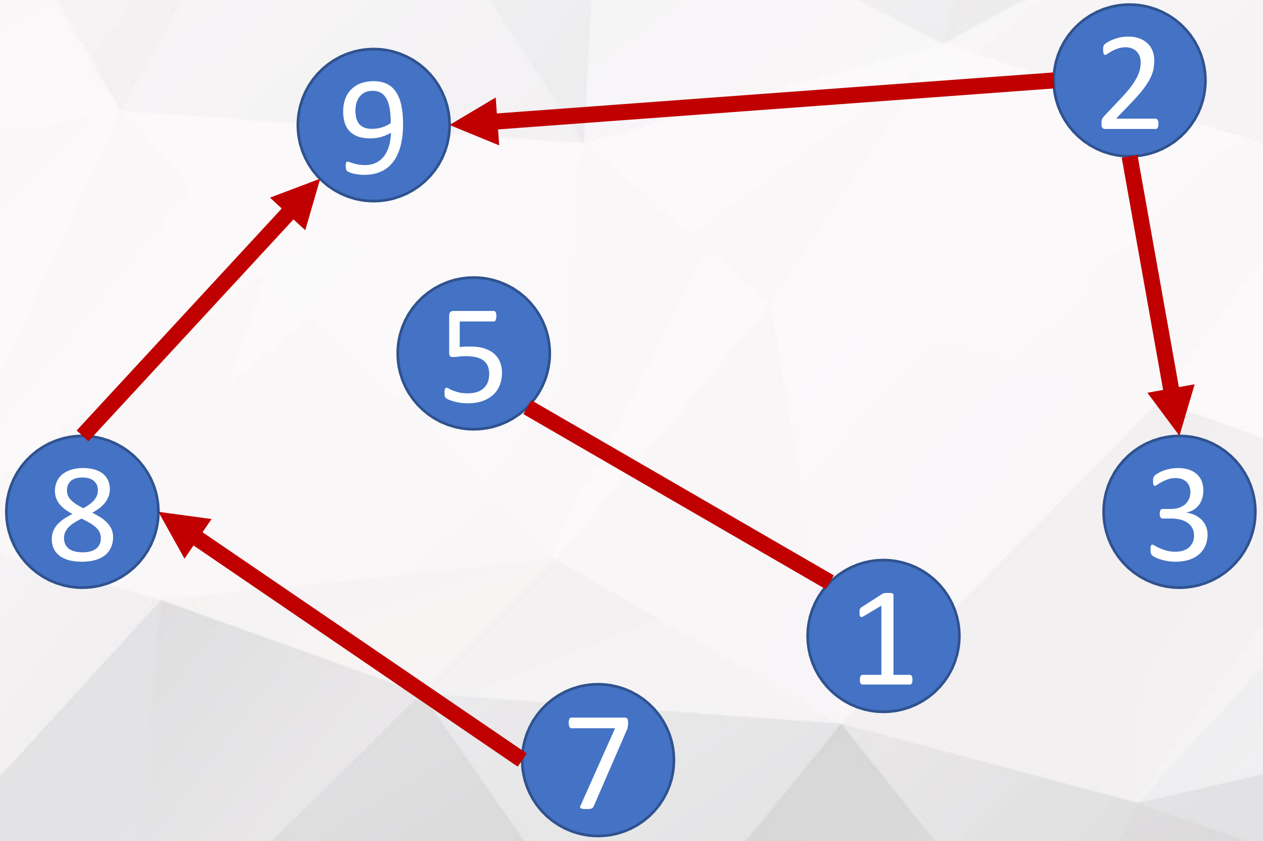
Graph 鄰接表

```
E[2][0] = 3;
E[8][0] = 9;
E[2][1] = 9;
E[7][0] = 8;

E[1][0] = 5;
E[5][0] = 1;
```



Graph 鄰接表

```
E[2][0] = 3;
E[8][0] = 9;
E[2][1] = 9;
E[7][0] = 8;

E[1][0] = 5;
E[5][0] = 1;
```



Tree

- Tree 是一個有向**無環**連通圖

Tree

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- 節點 (node)： 一般樹上的點

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- 節點 (node)： 一般樹上的點
- 父 (parent)： 節點能**反向**拜訪的**第一**個節點
- 子 (child)： 節點能**正向**拜訪的**第一**個節點

Tree

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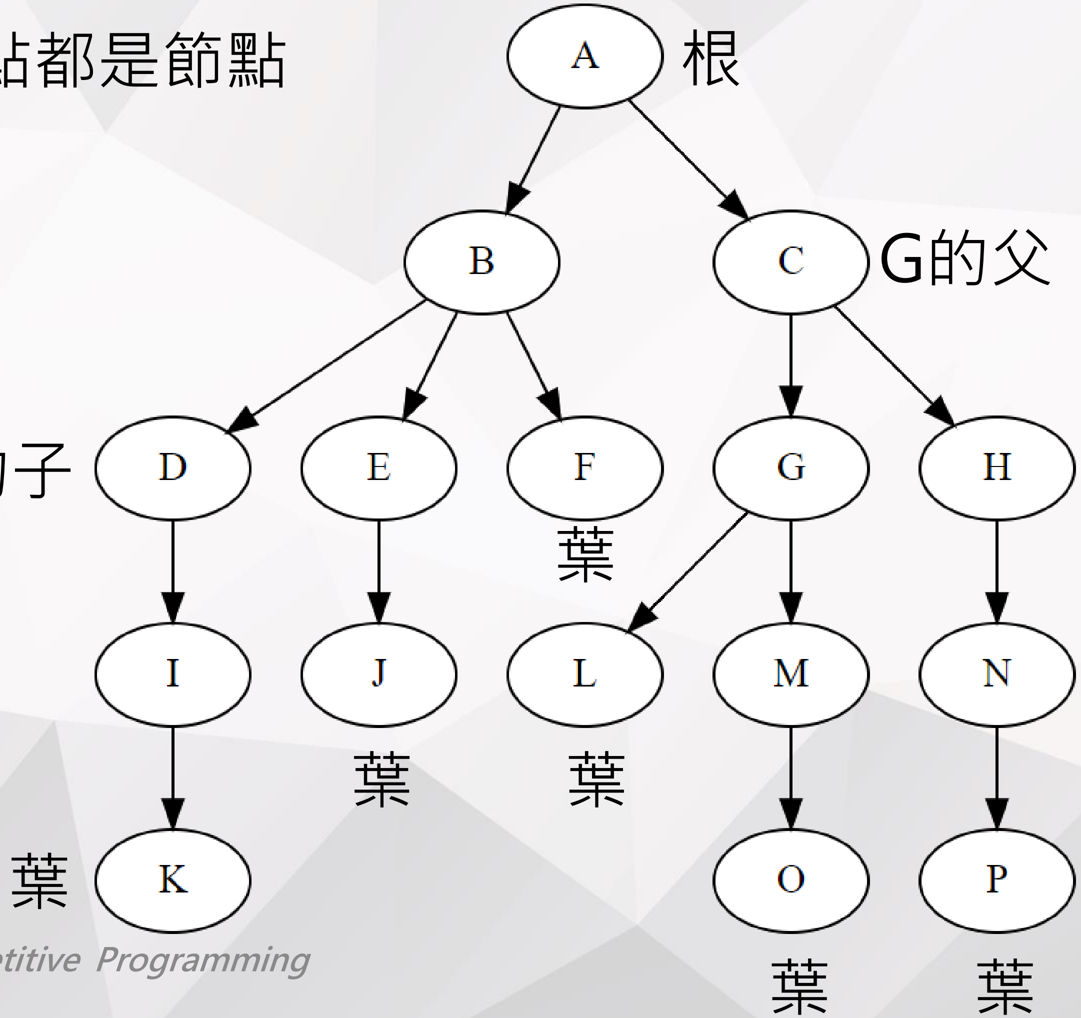
- 節點 (node)： 一般樹上的點
- 父 (parent)： 節點能**反向**拜訪的**第一**個節點
- 子 (child)： 節點能**正向**拜訪的**第一**個節點
- 根 (root)： 沒有父節點的節點
- 葉 (leaf)： 沒有子節點的節點

Tree

每個點都是節點



Tree

- 祖先 (ancestor)： 節點能**反向**拜訪的所有節點
- 孫子 (descendant)： 節點能**正向**拜訪的所有節點

Tree

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- 深度 (depth)： 從**根**到該節點所**經過的邊數**

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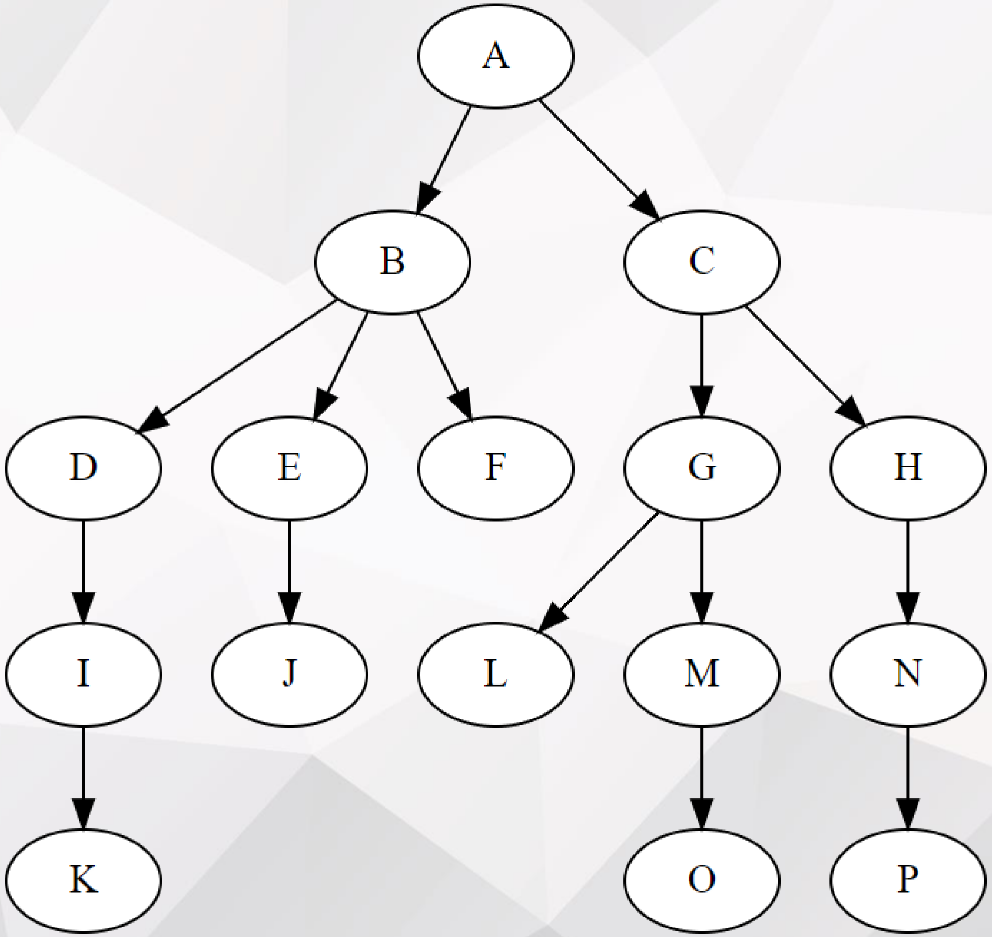
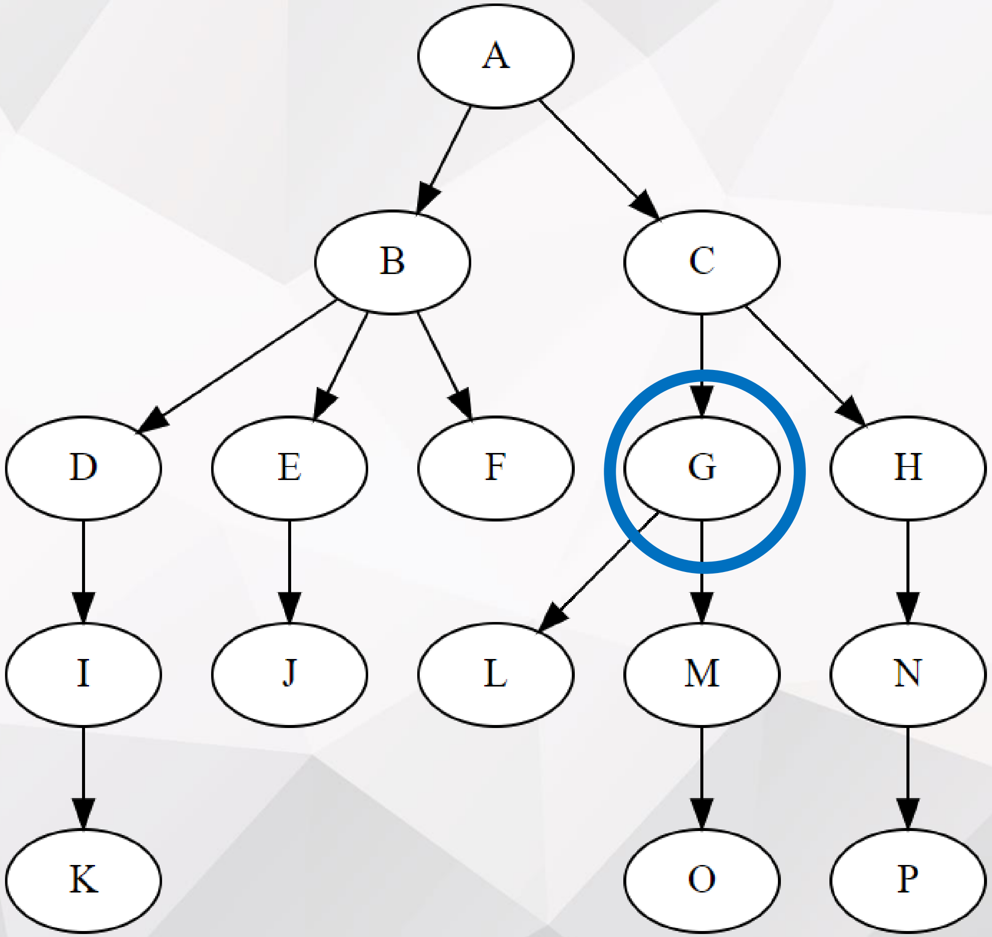
- 森林 (forest)： 一個集合包含所有**不相交**的 Tree
- 每個非根節點**只有一個**父節點

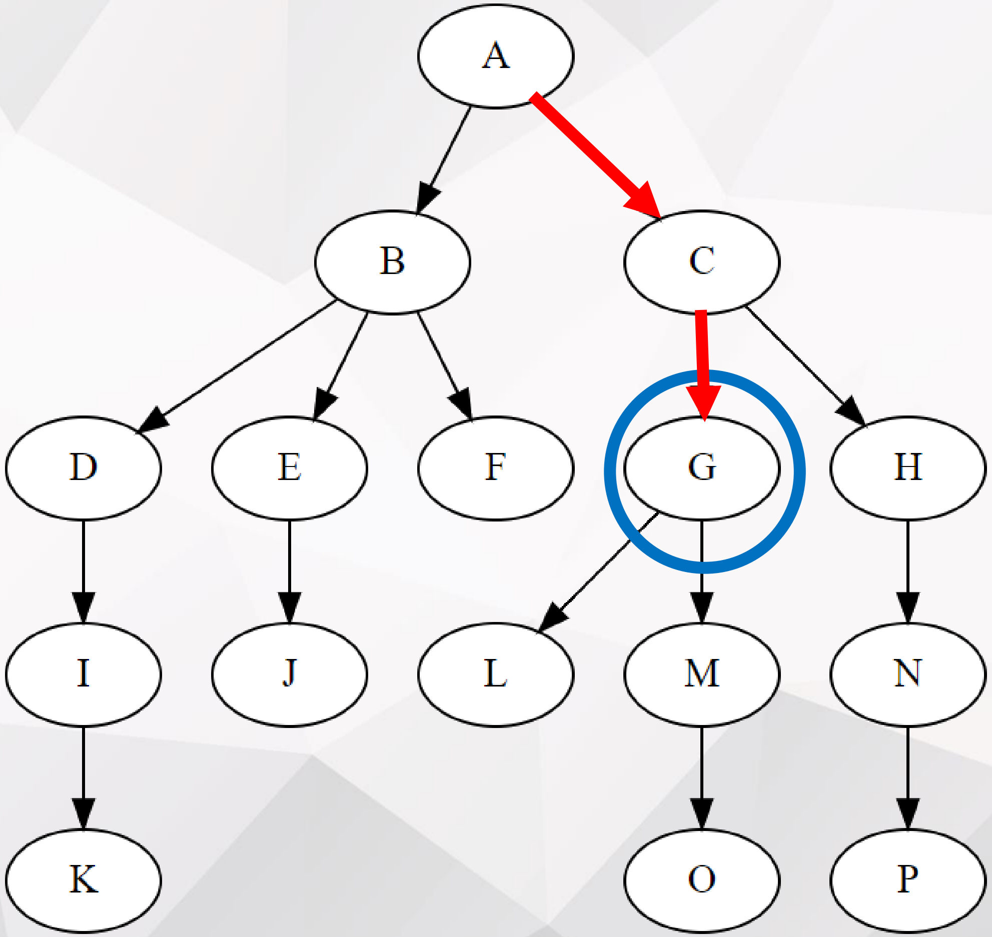
Tree



Tree

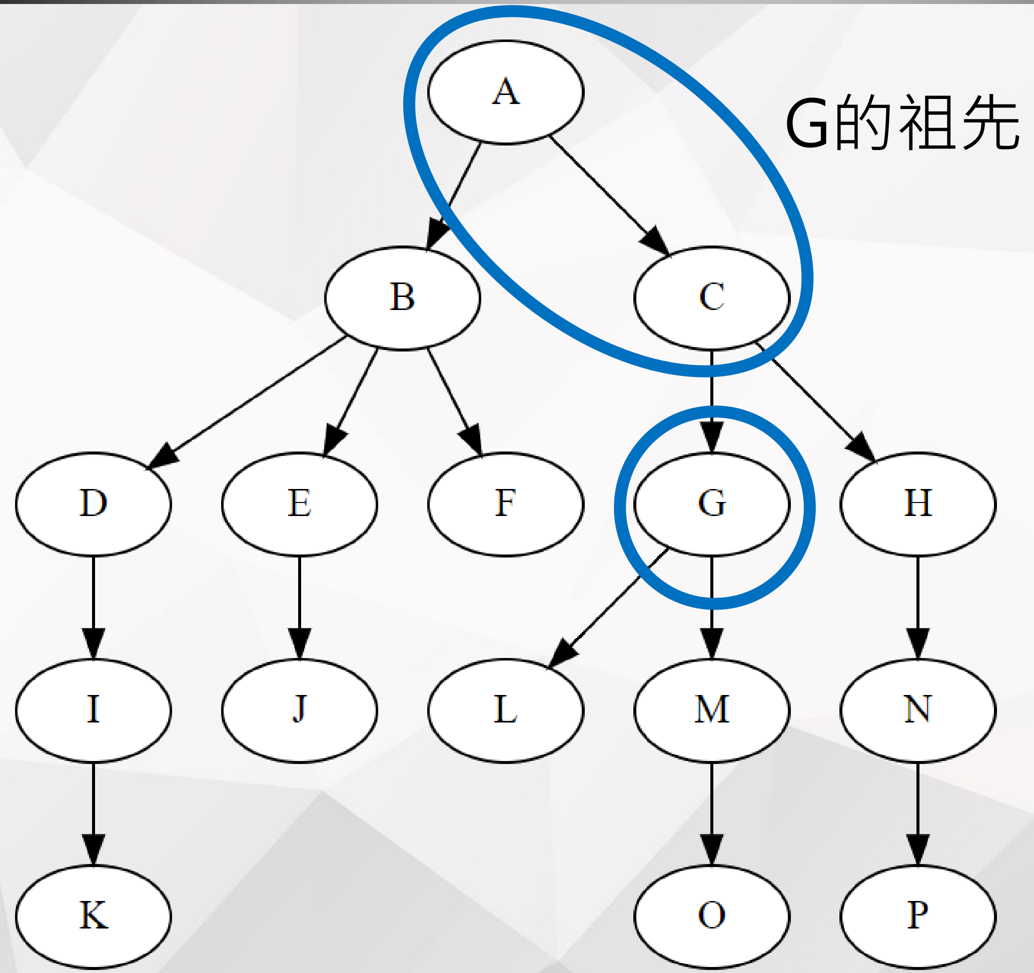


Tree

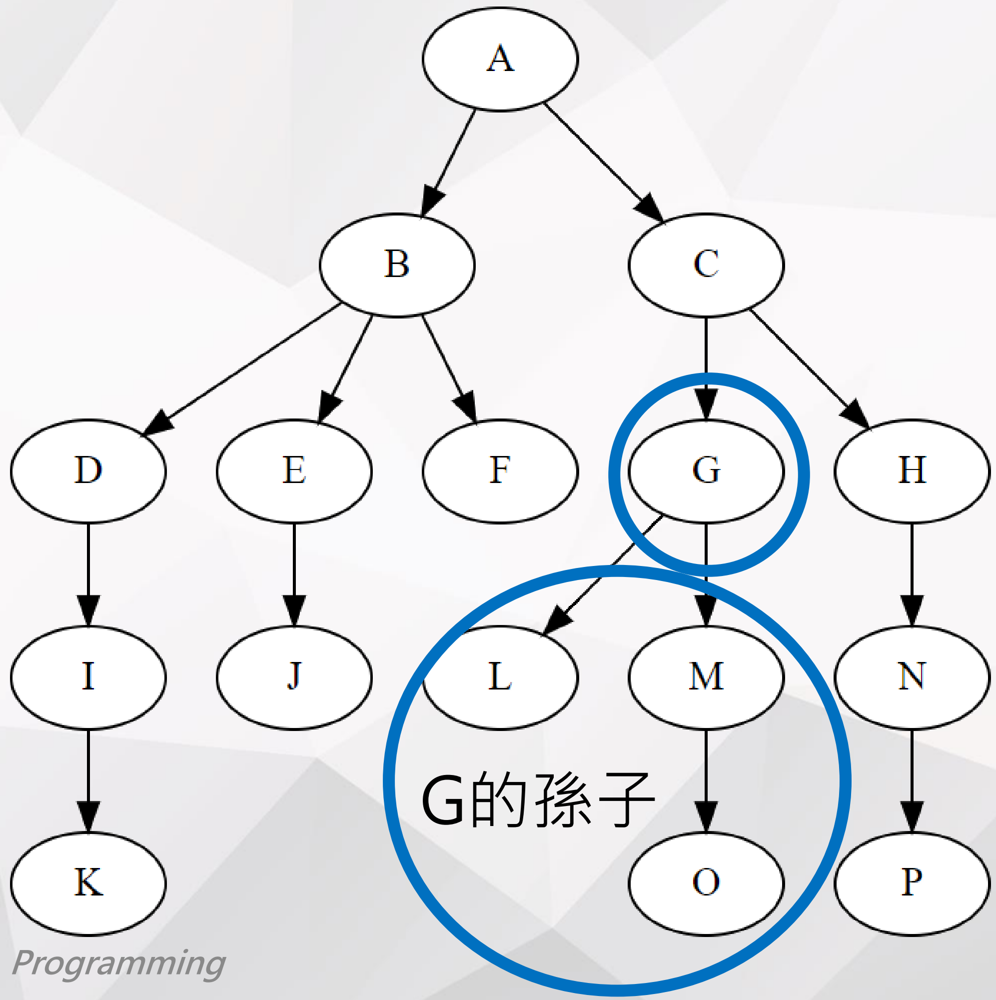


G的Depth為2

Tree



Tree



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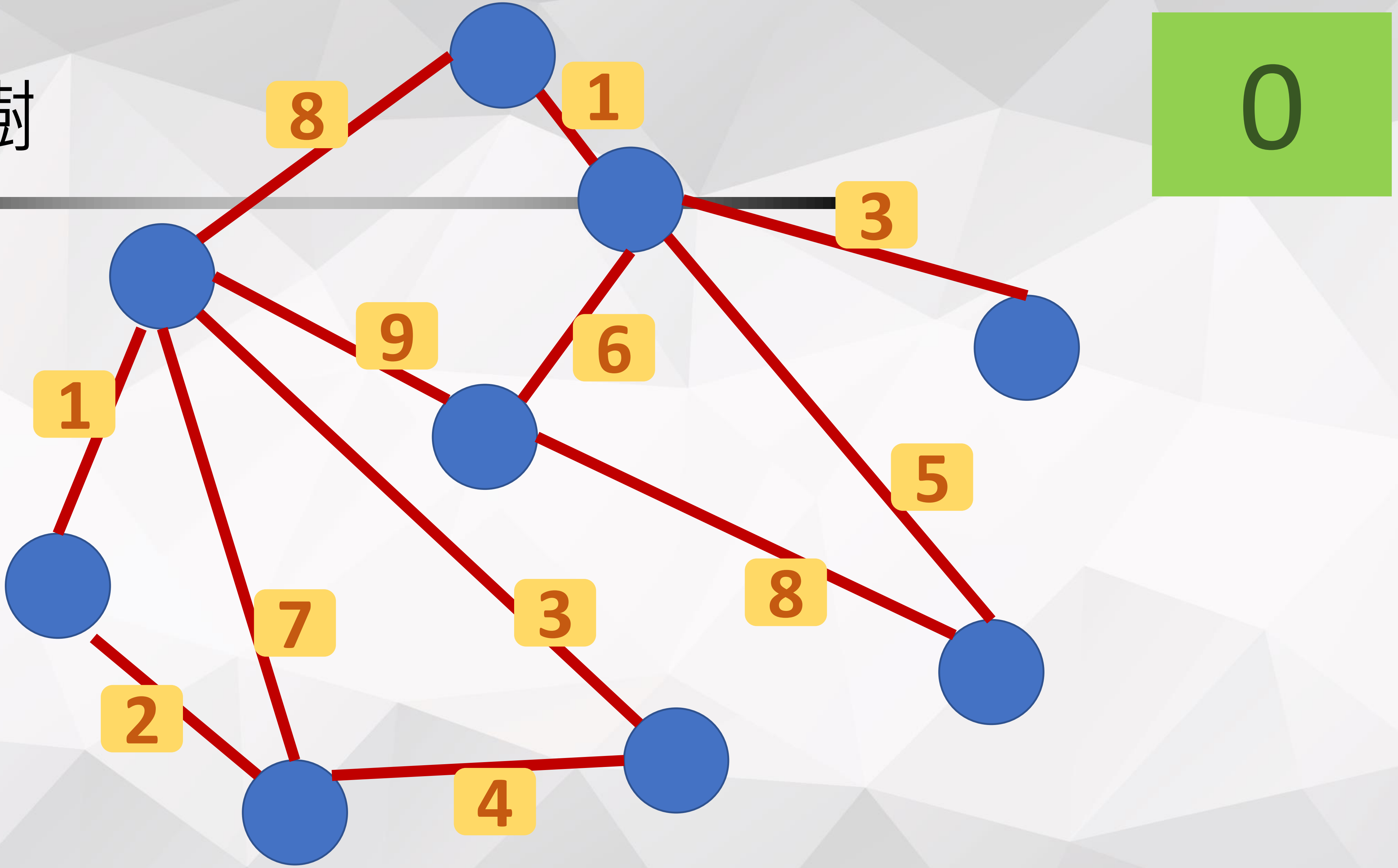
Minimum spanning tree

生成樹

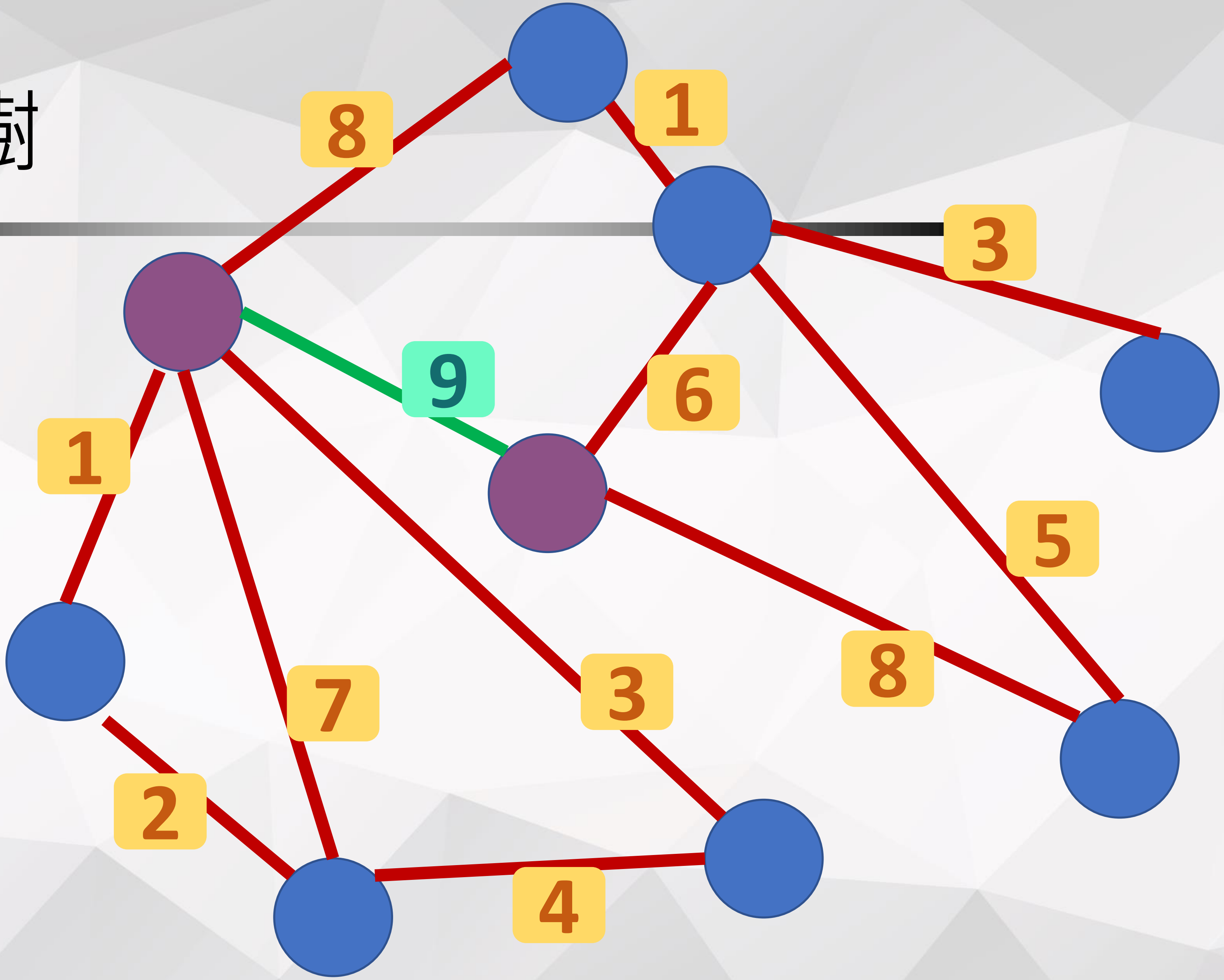
- 給定**連通圖** G = (E, V)
- 使用 E 子集**連接所有的點** (屬於 V) 所得到的**樹**

生成樹



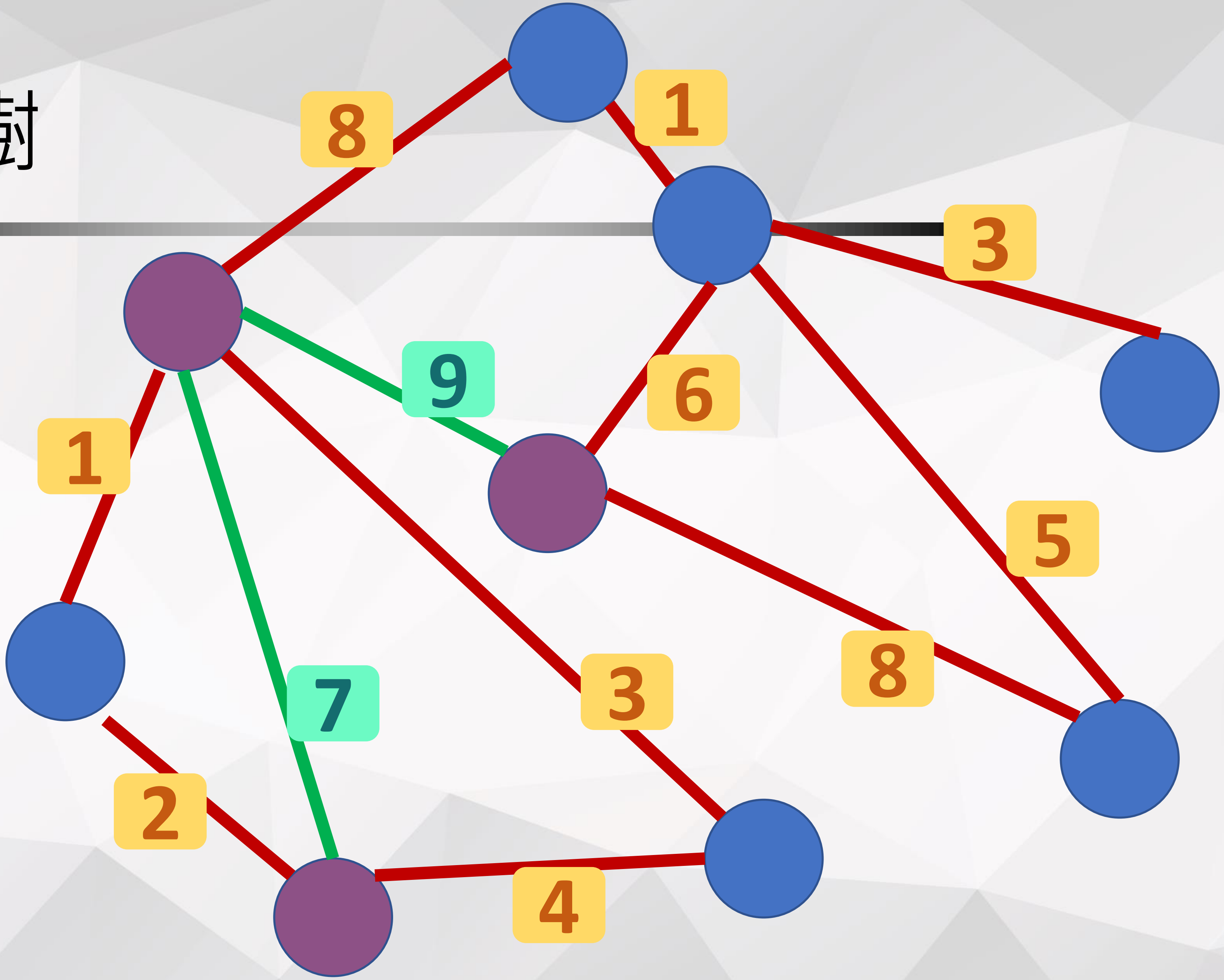
生成樹

9

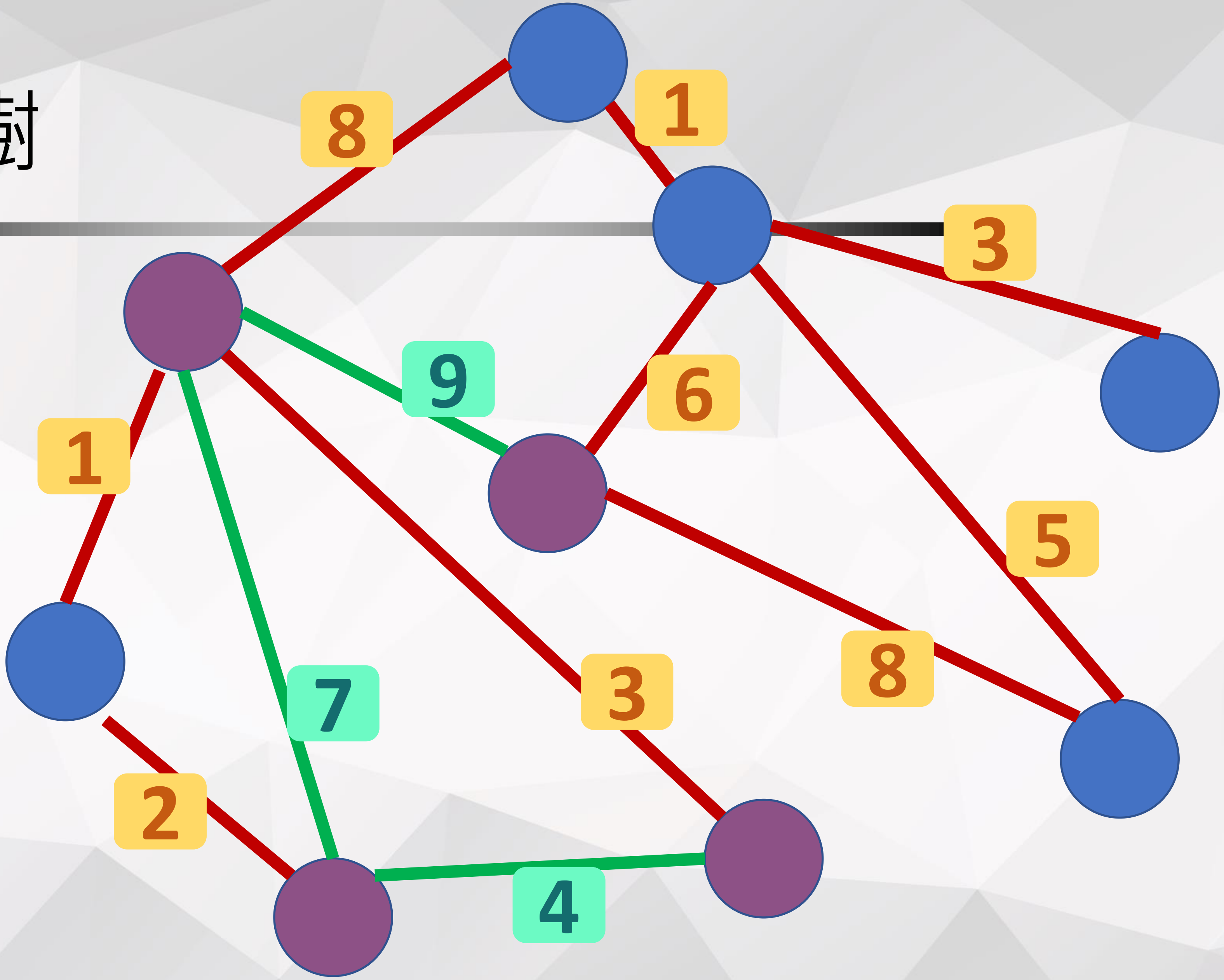


生成樹

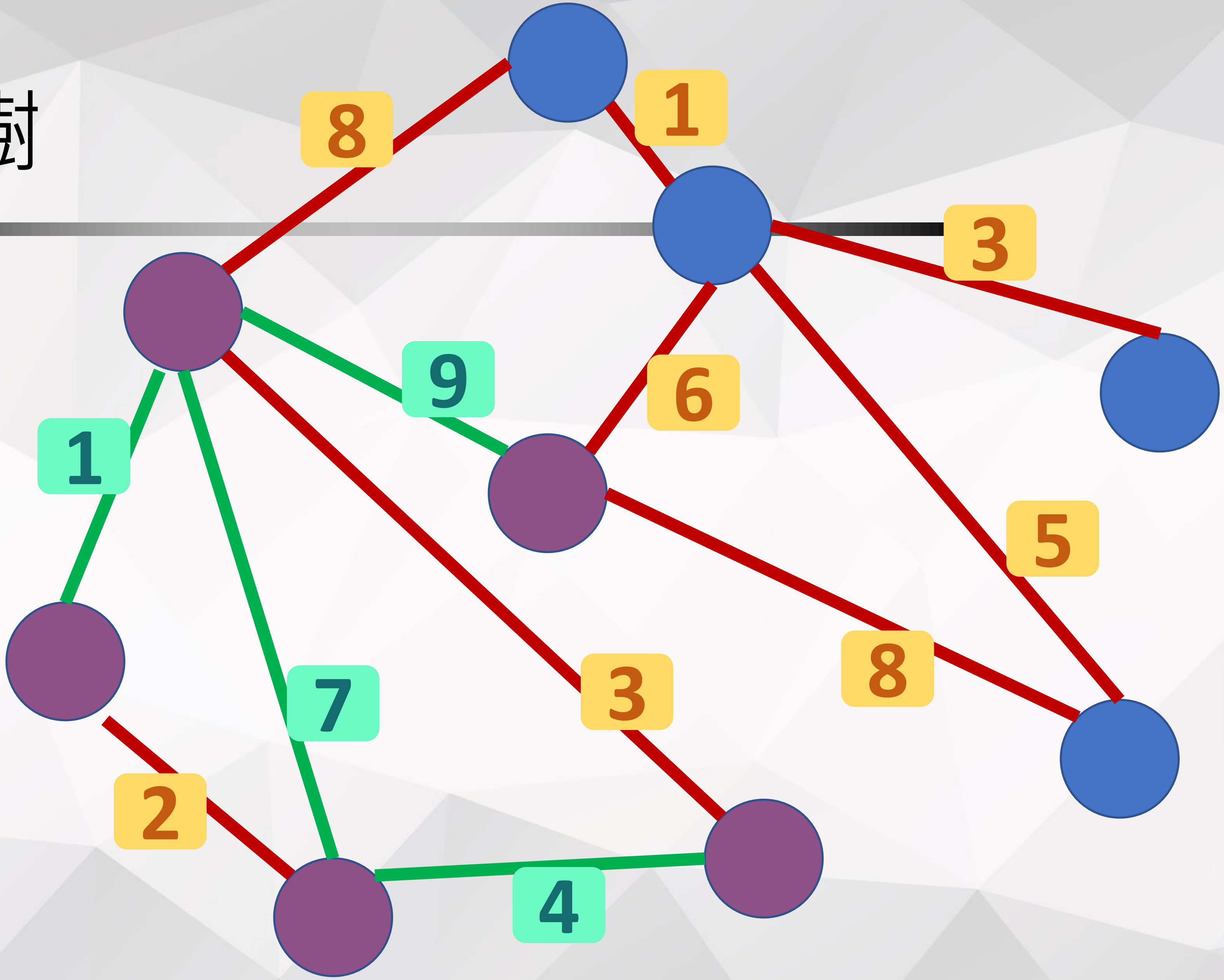
16



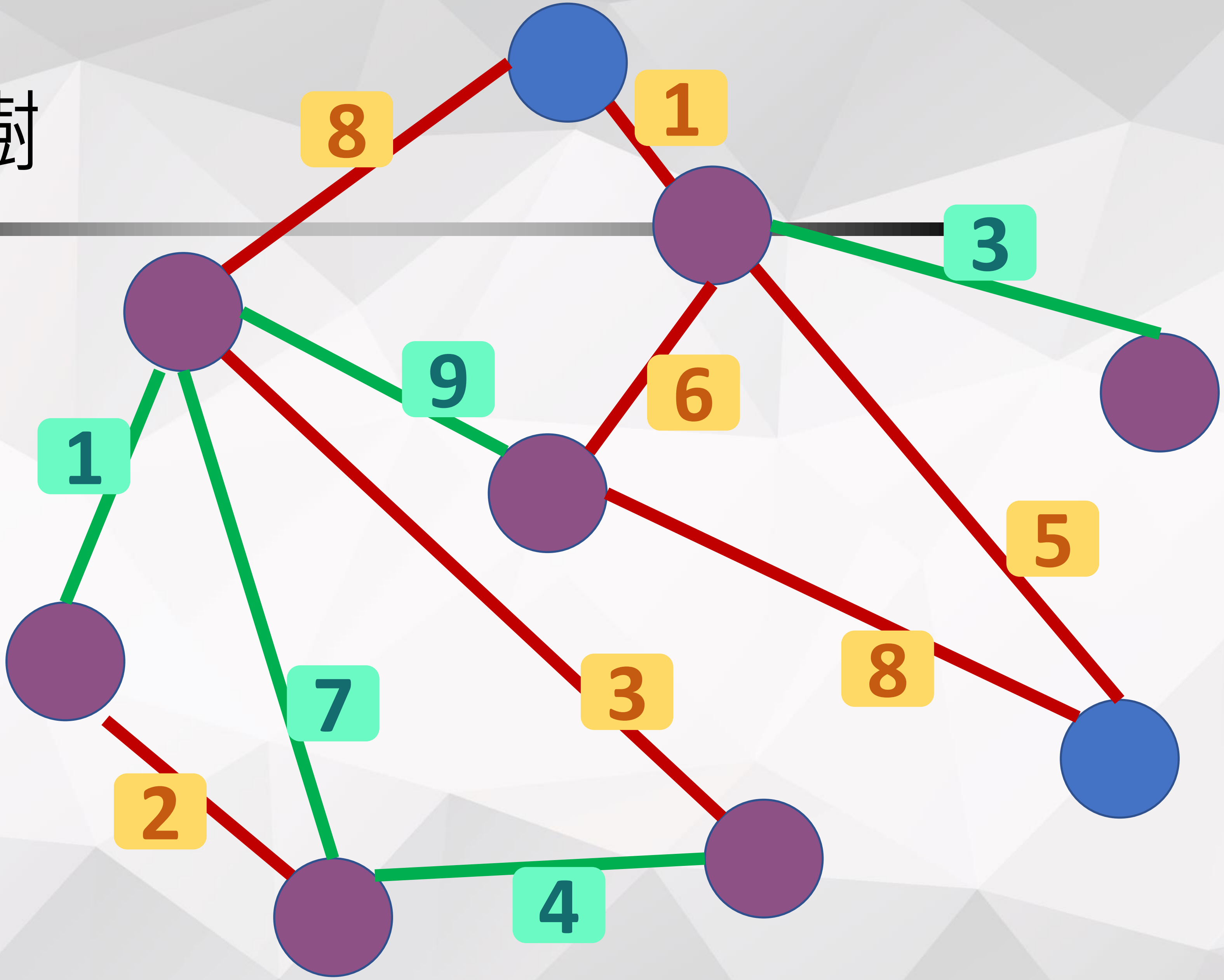
生成樹



生成樹

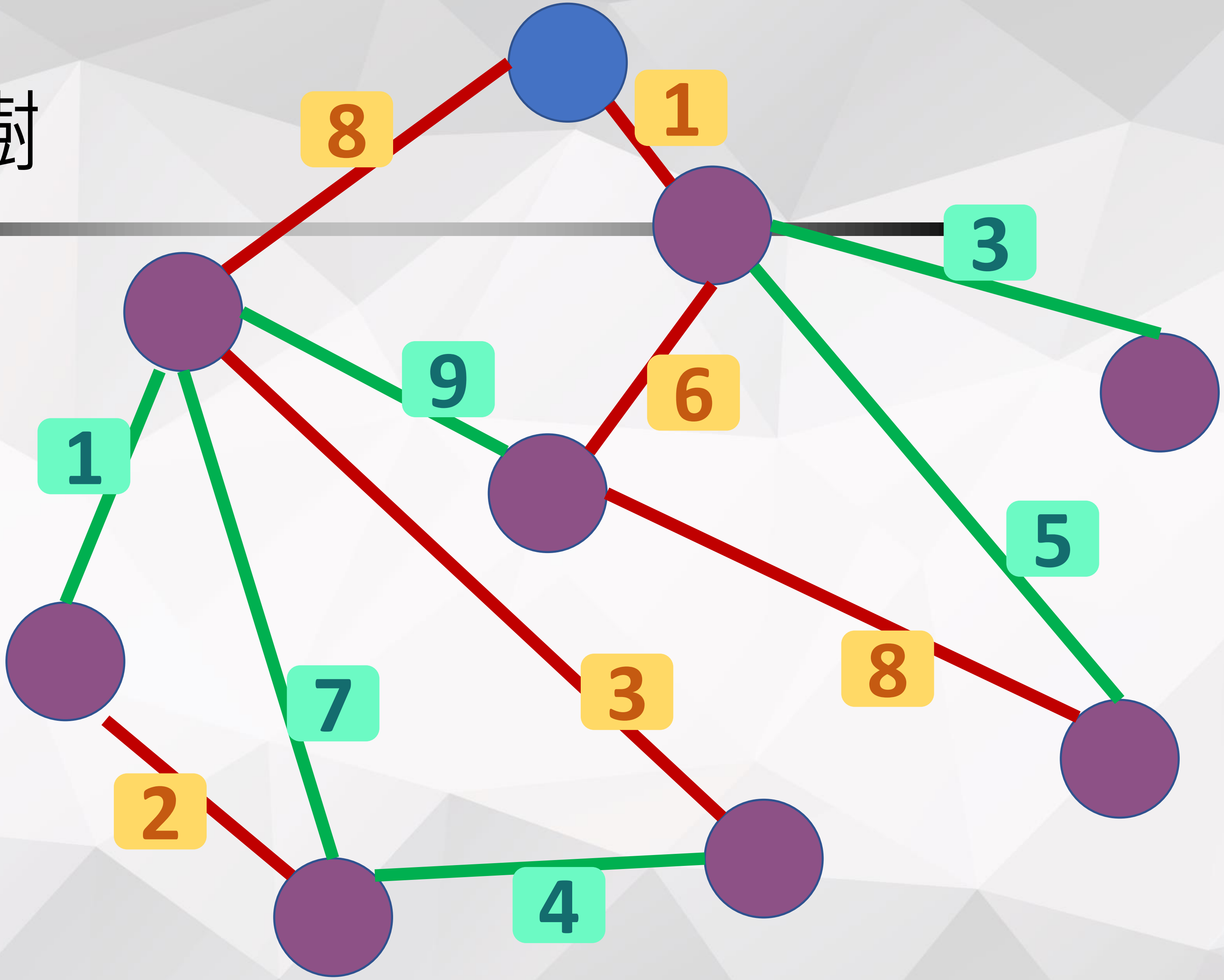


生成樹

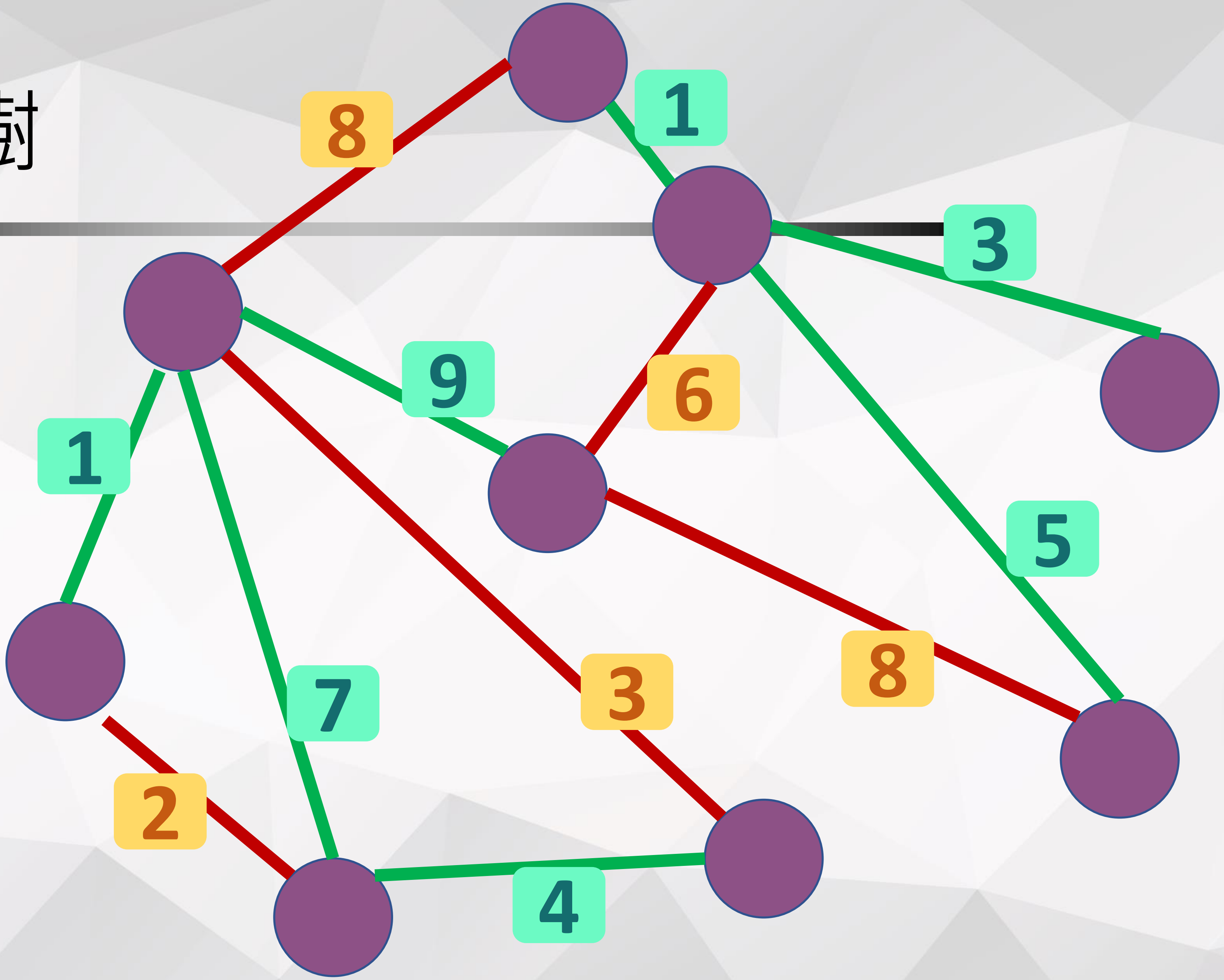


生成樹

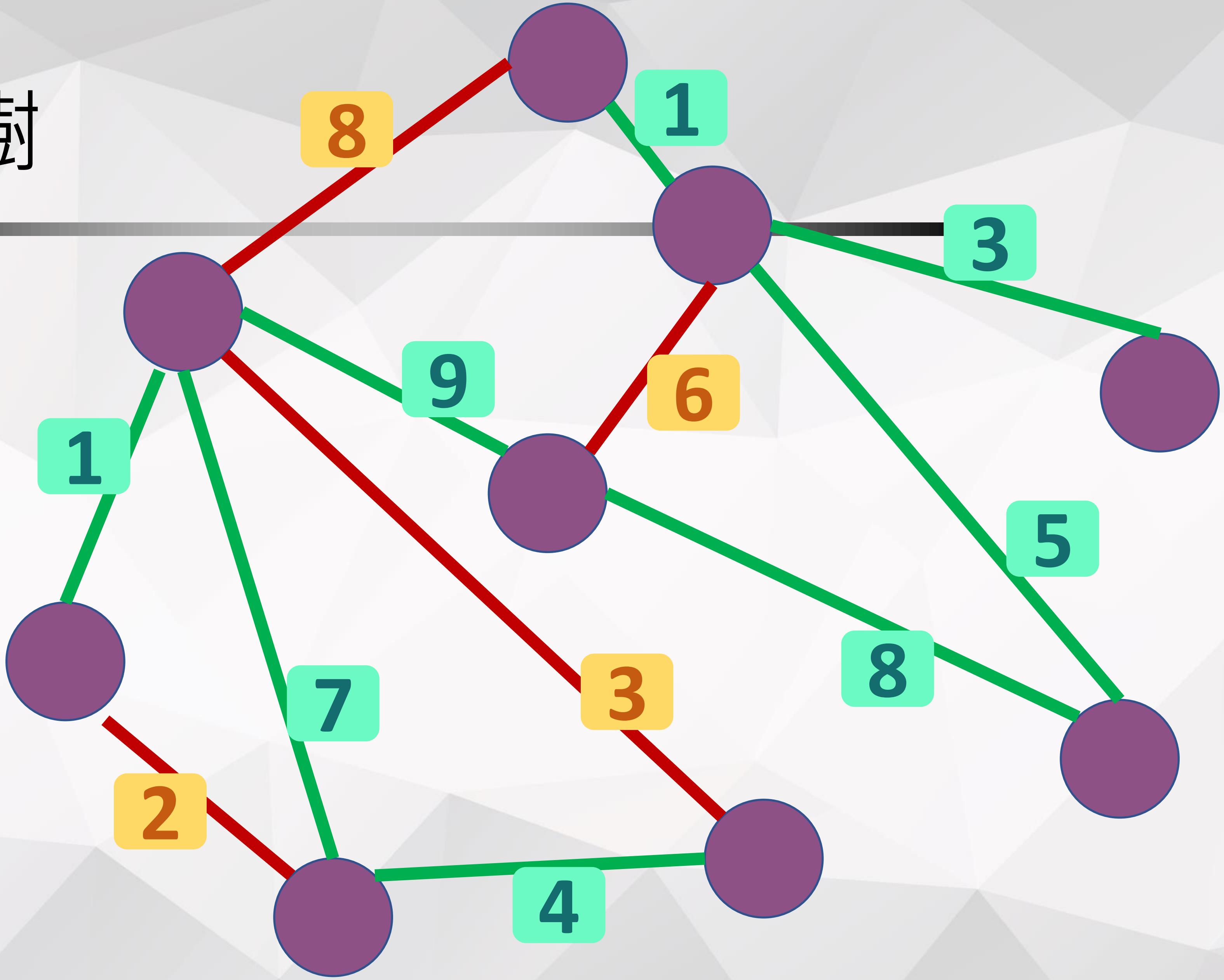
29



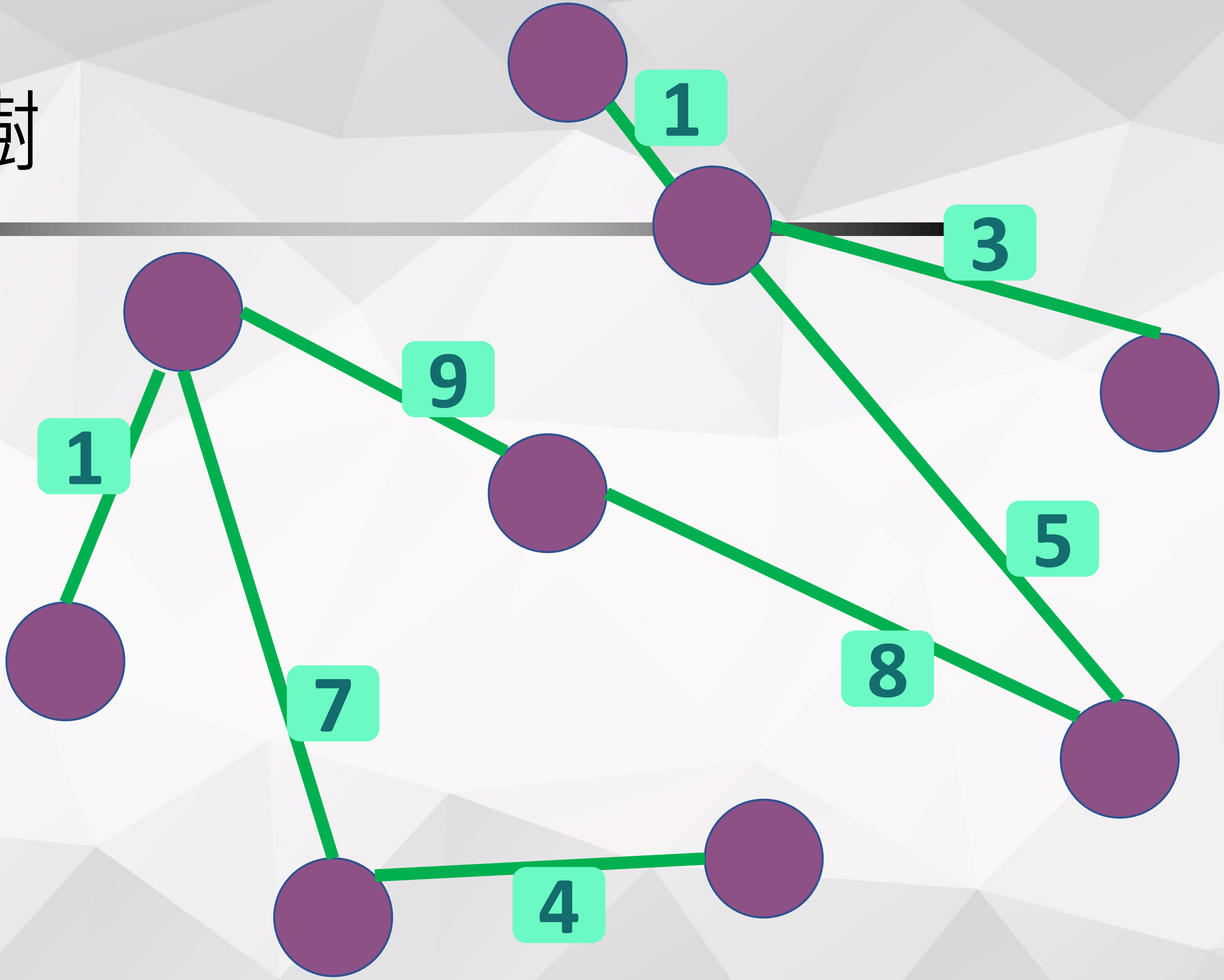
生成樹



生成樹



生成樹

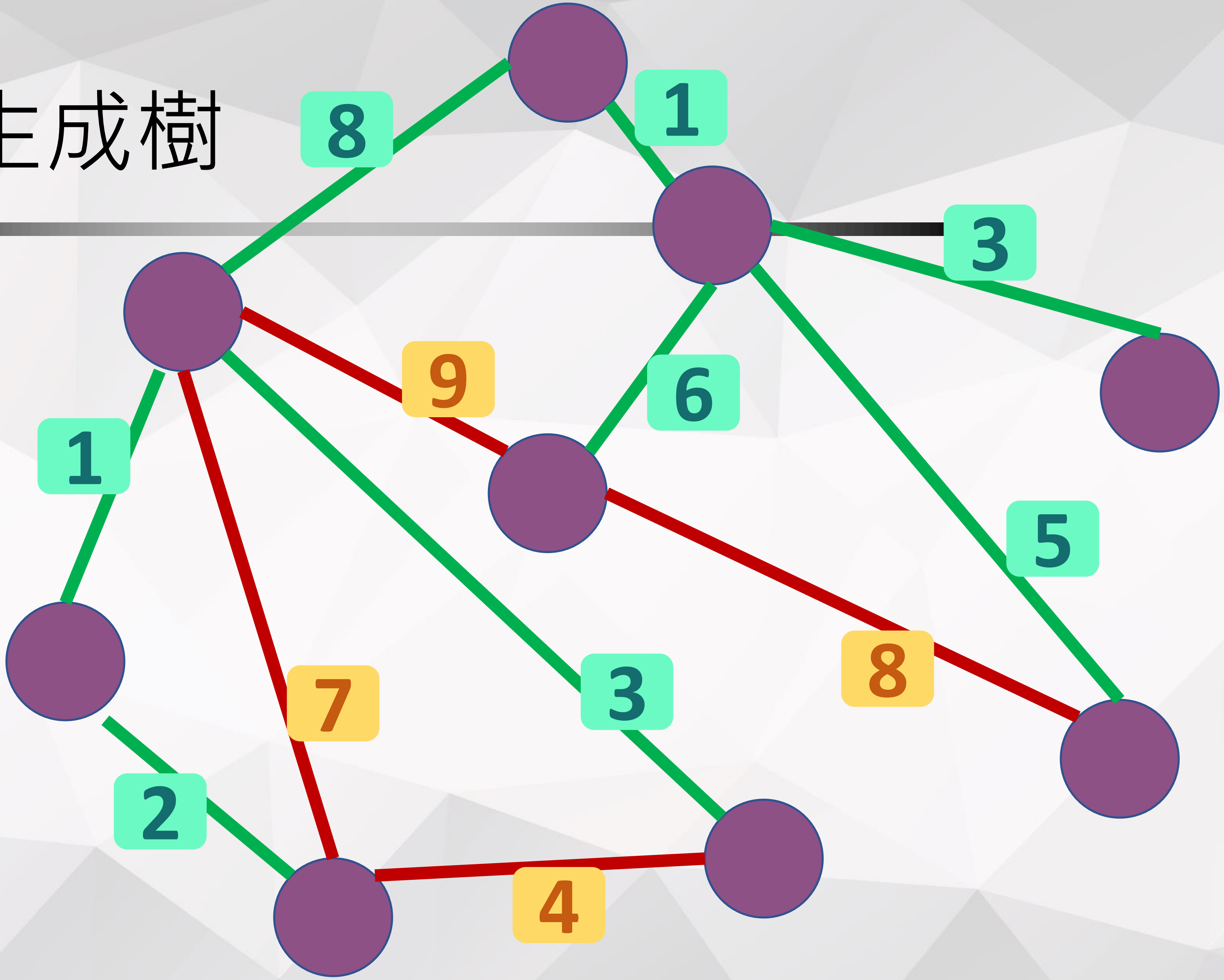


最小生成樹

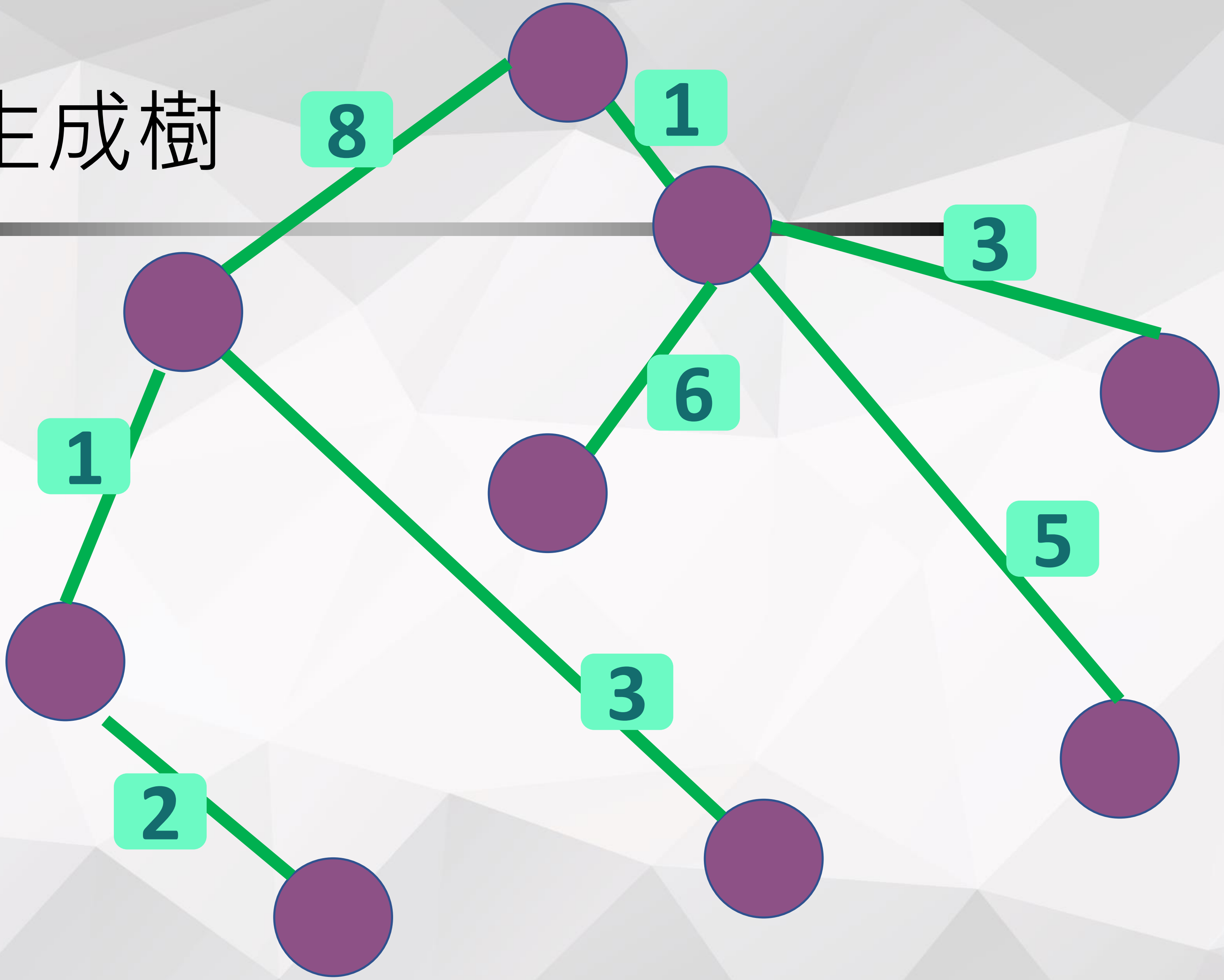
- 給定連通圖 G = (E, V)

- 在所有生成樹中，找到**邊權重總和最小**的生成樹

最小生成樹



最小生成樹

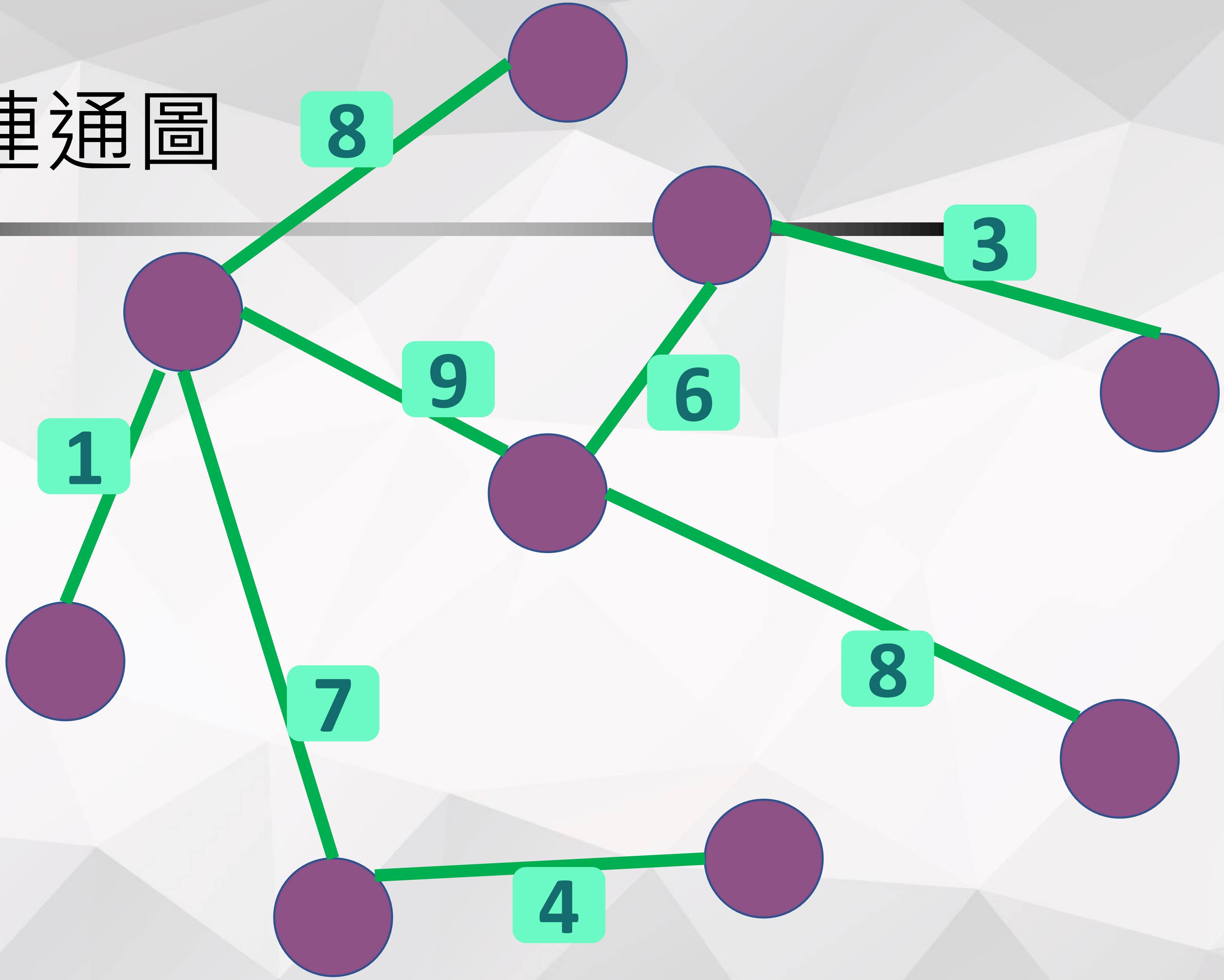


兩個重要前提

- 樹是**無環**的**連通**圖

- 若圖只有點無任何邊，那每點都是彼此獨立**連通塊**

無環連通圖



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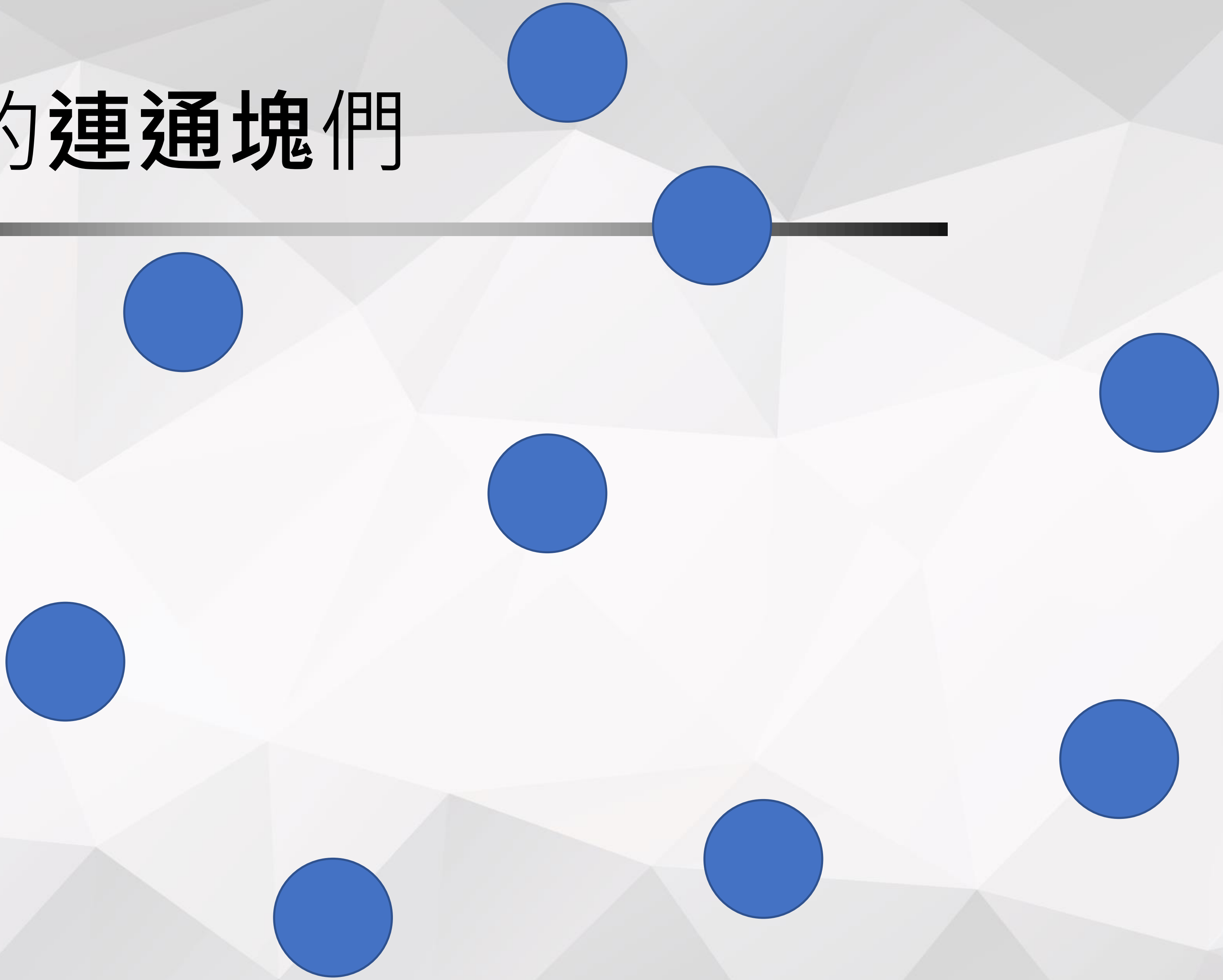
獨立的**連通塊**們



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最小生成樹

- Kruskal 演算法
- Prim 演算法

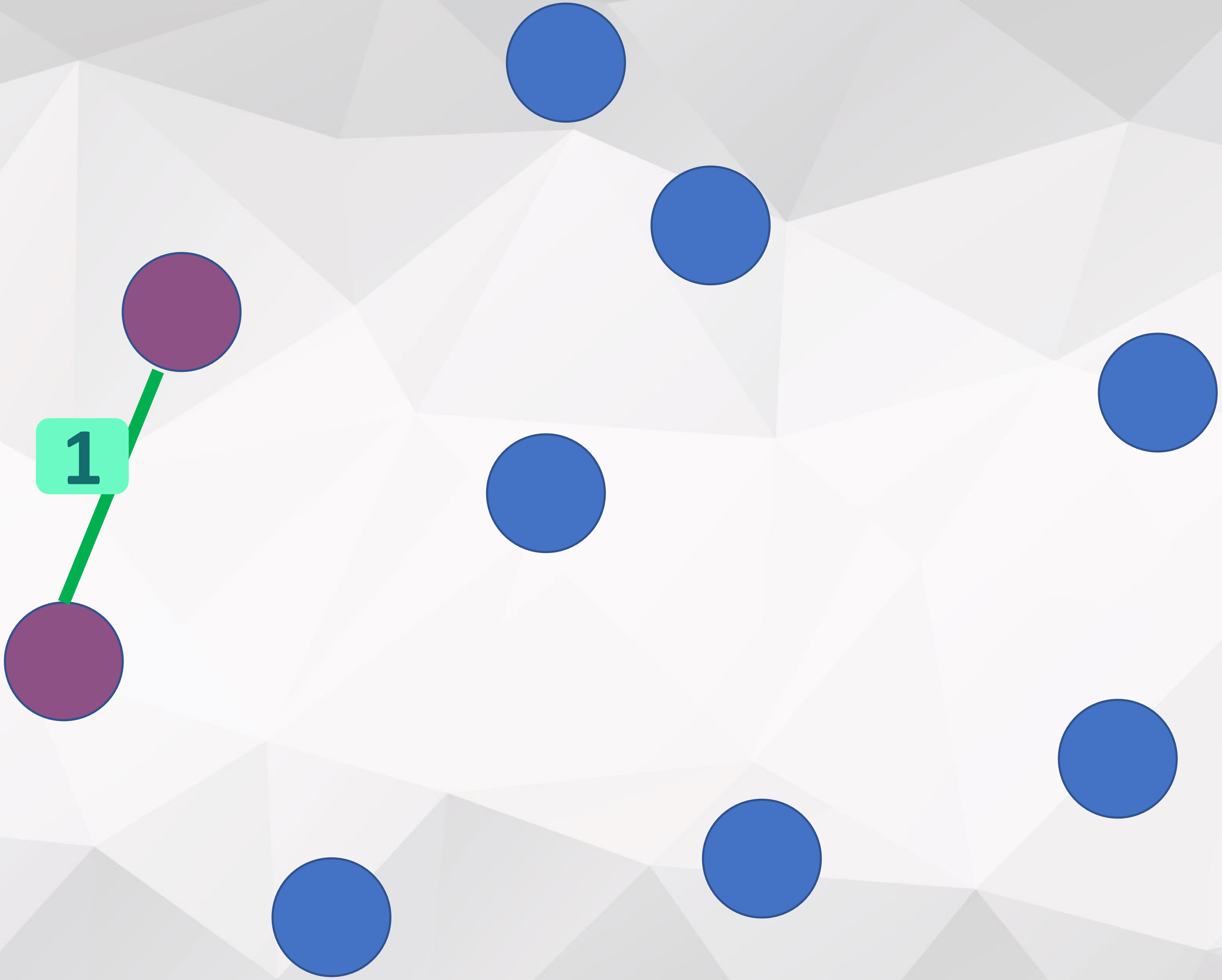
最小生成樹

- Kruskal 演算法
- Prim 演算法

Kruskal 演算法

- 在產生生成樹以前，所有點都為獨立的連通塊
- 若兩個獨立的連通塊相連，整個圖就少一連通塊

0

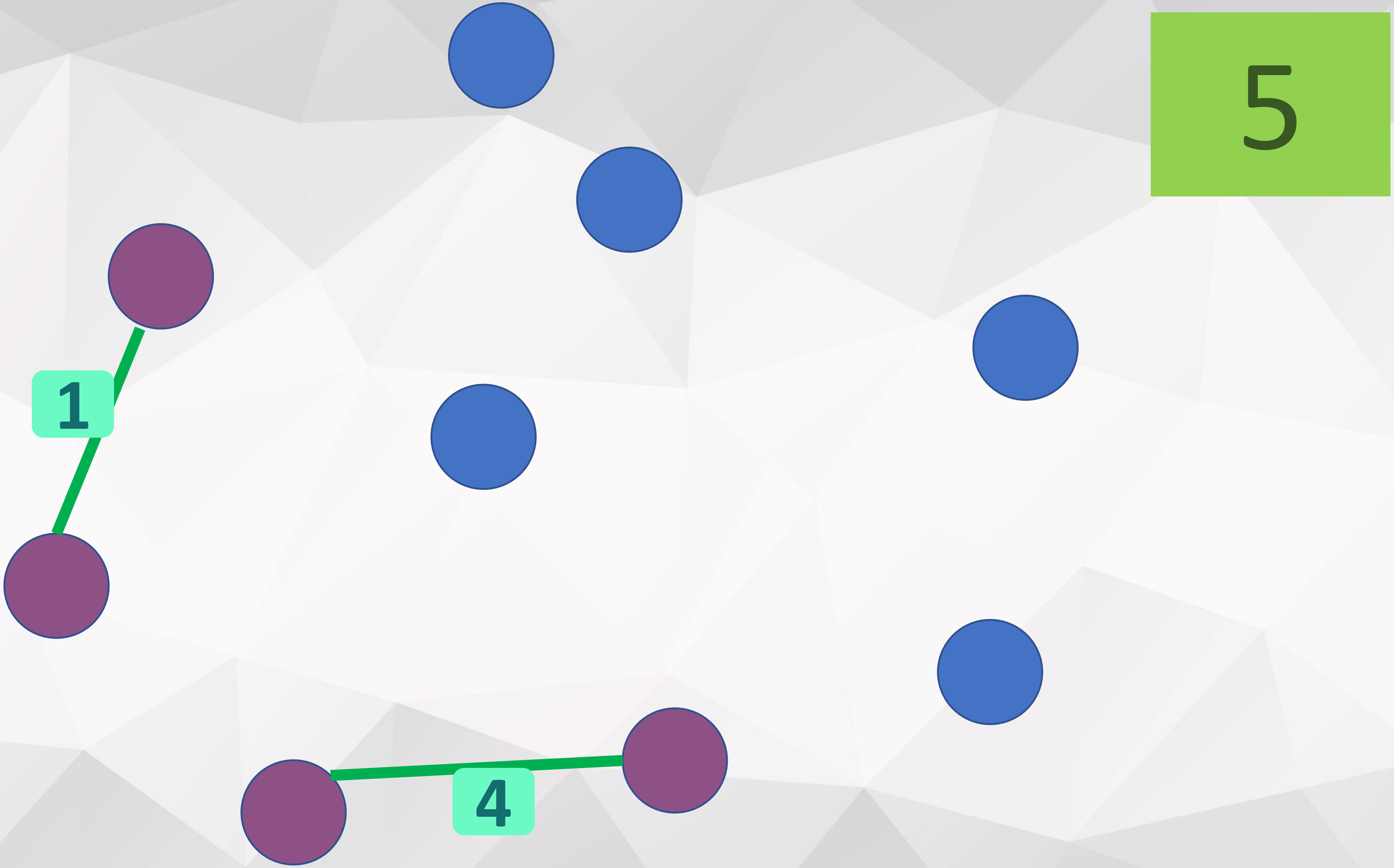


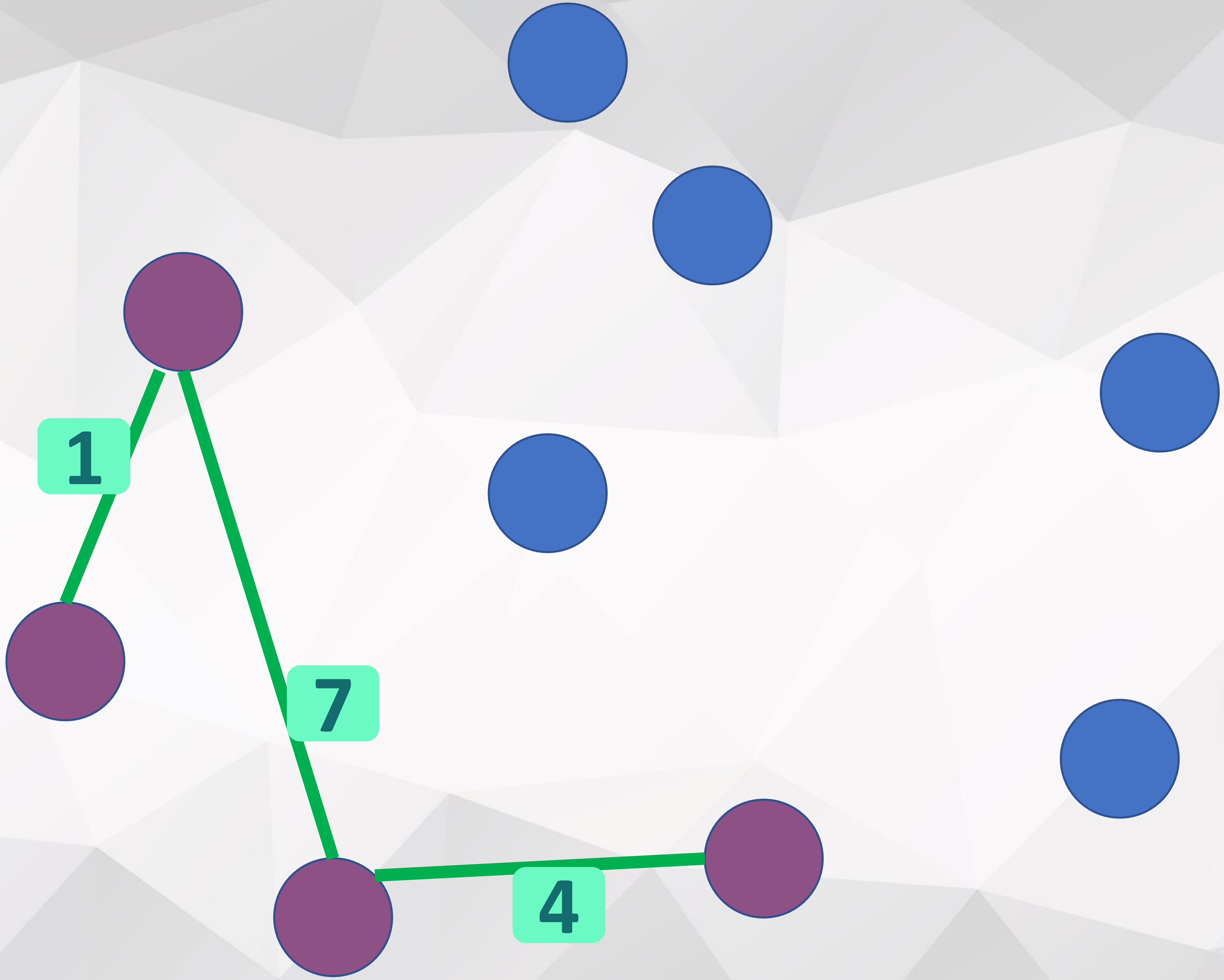
1

如何產生生成樹

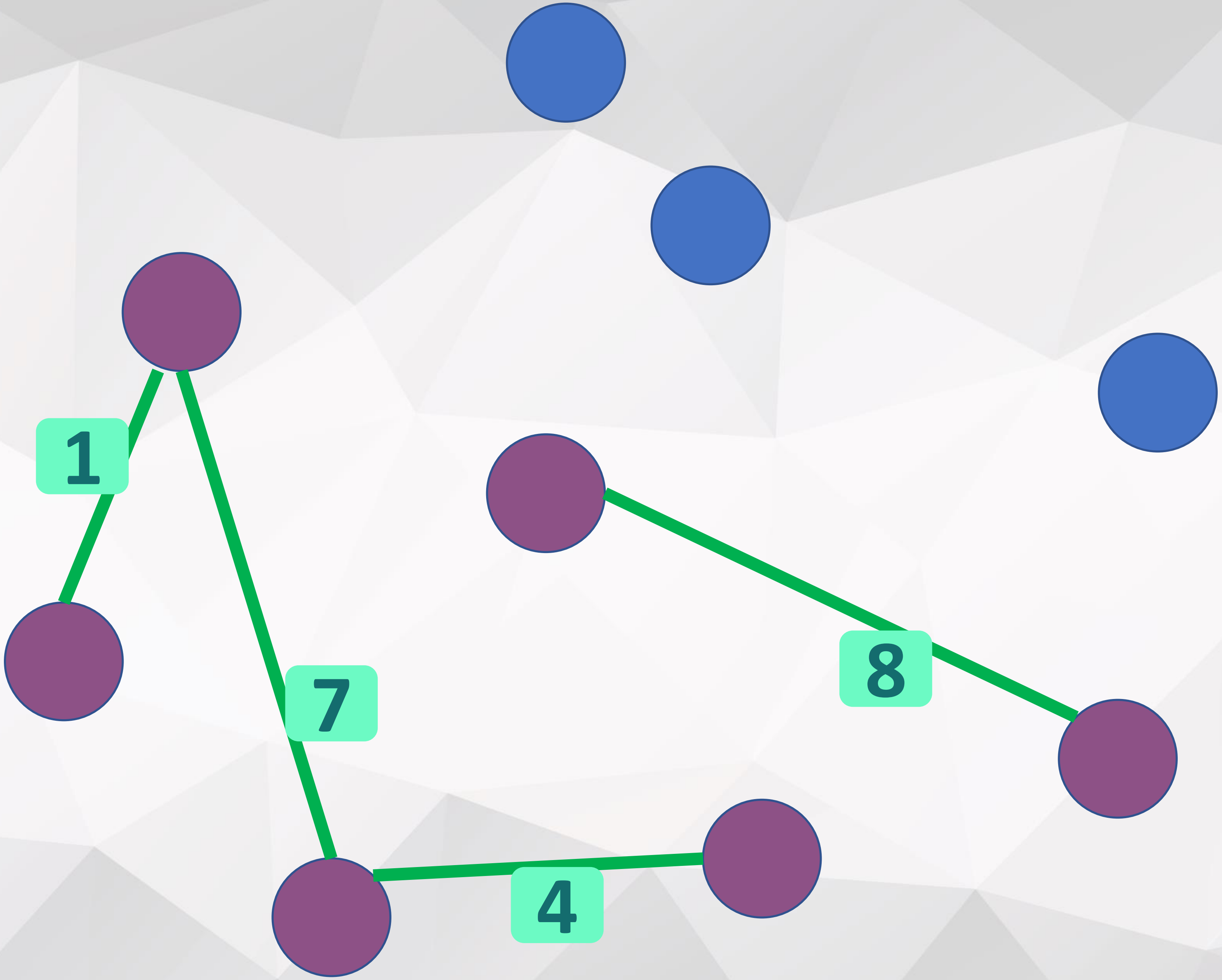
- 在連通塊 A 與連通塊 B 相連時確保不會產生環
- 最終就能得到一棵生成樹

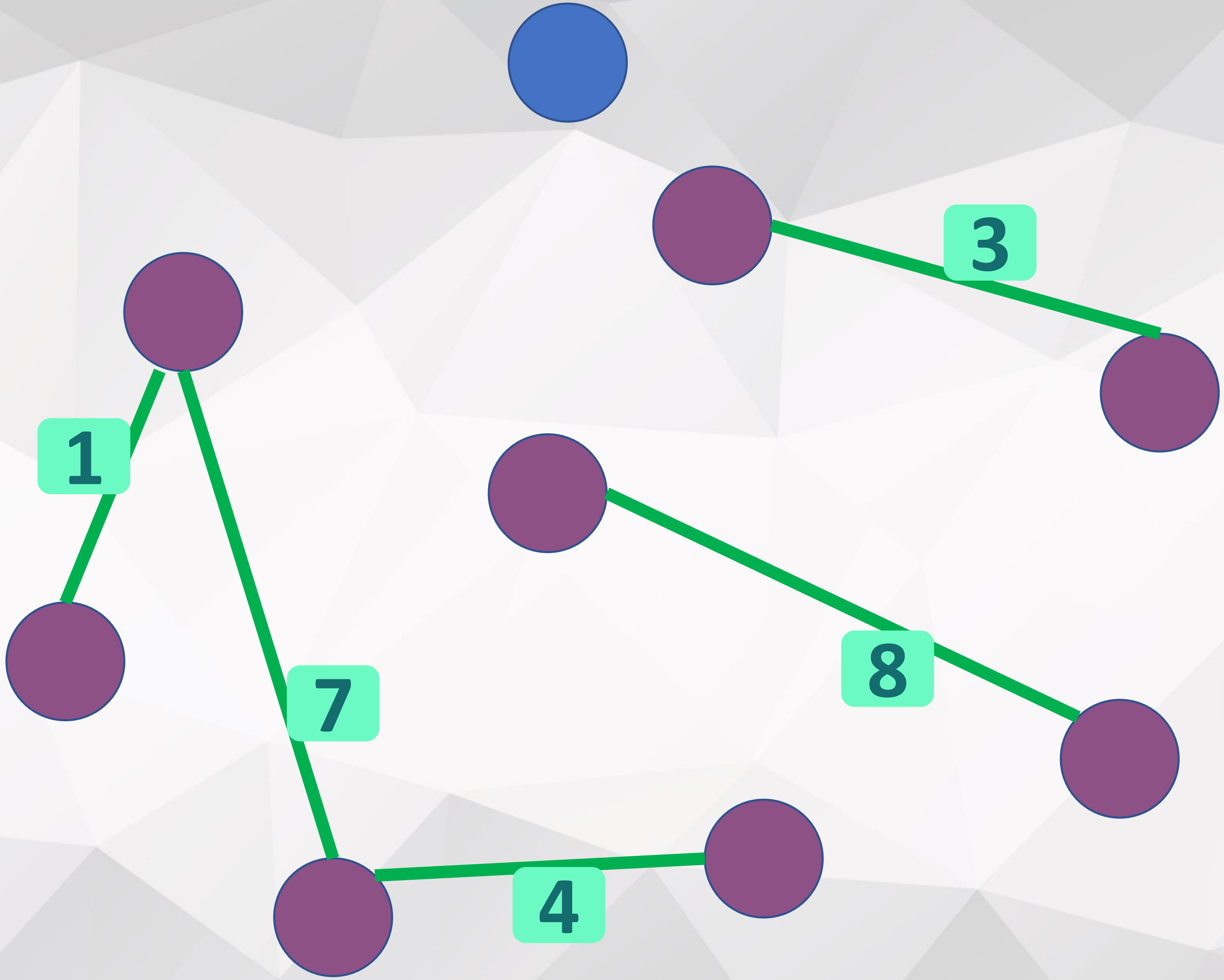
1
1

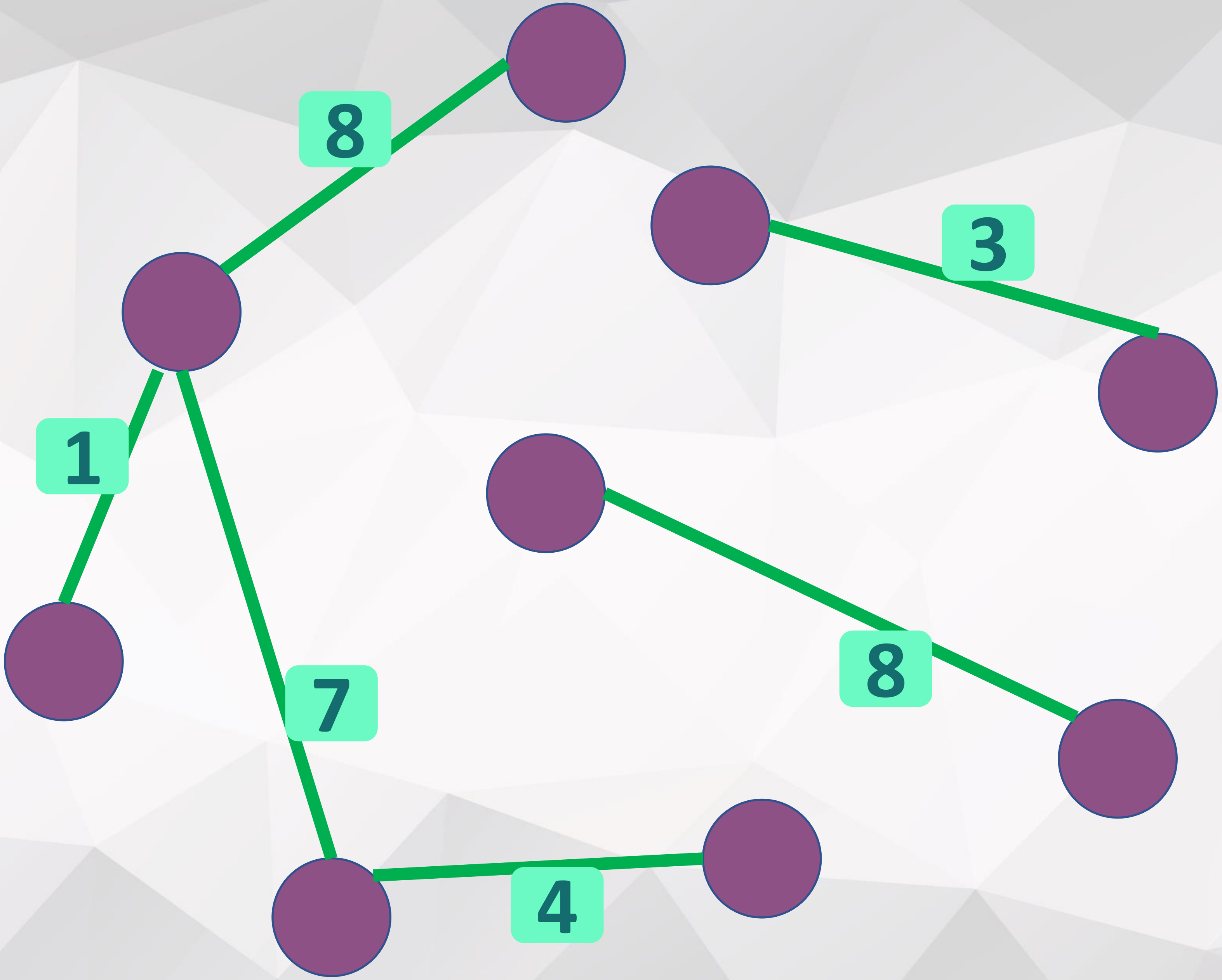

5
1
4

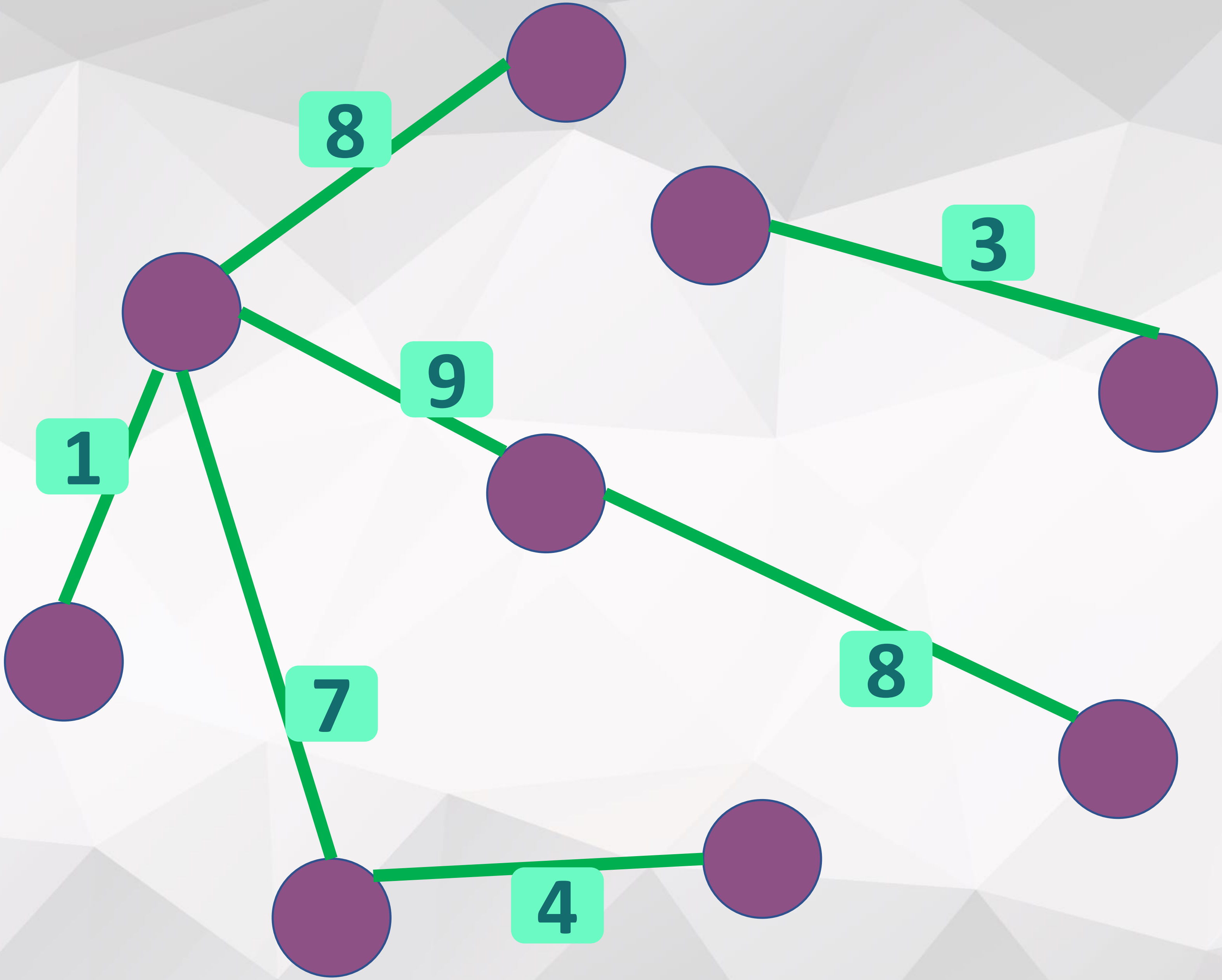

1
7
4

20




3
1
7
8
4


8
3
1
7
8
4



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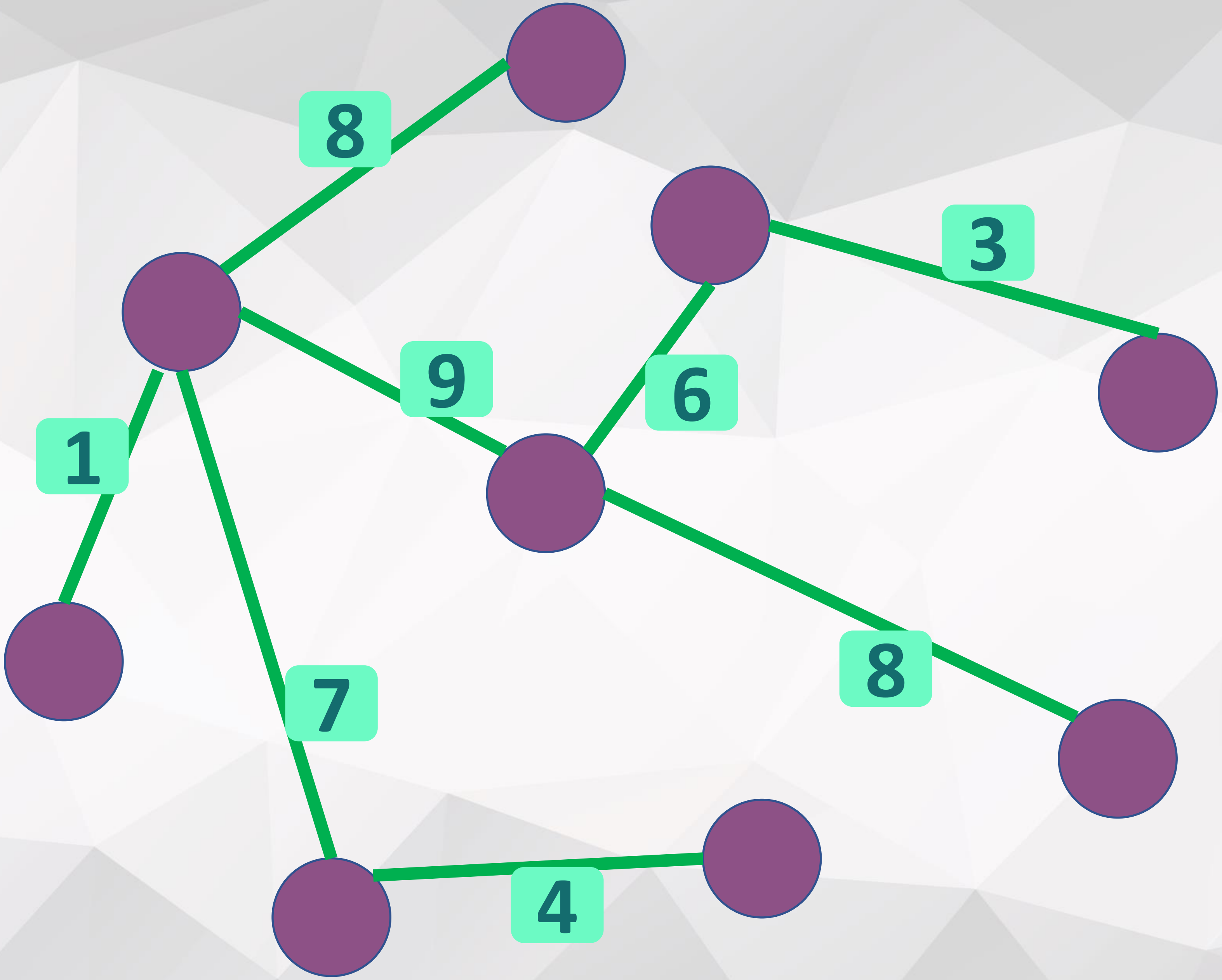

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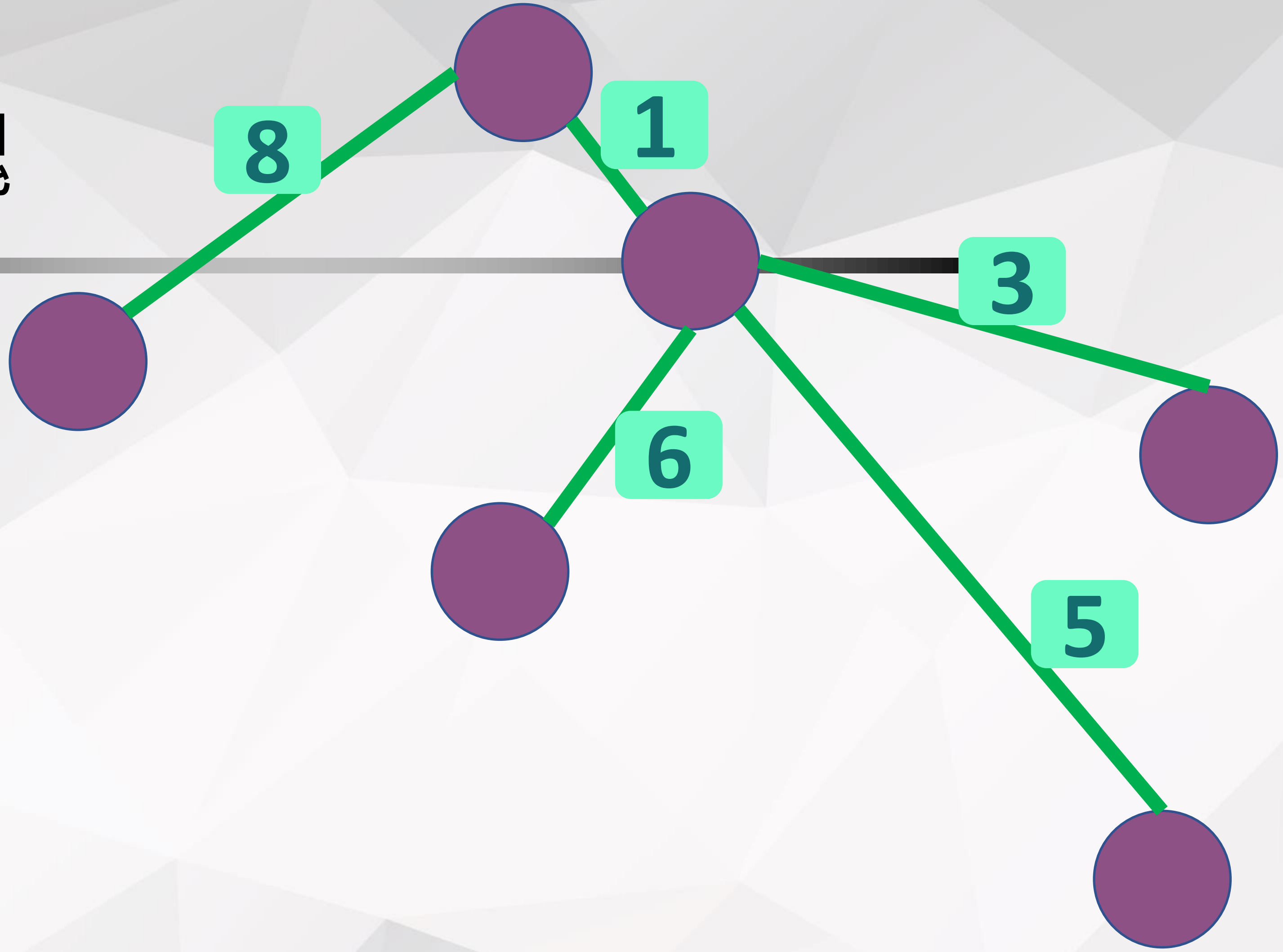
Competitive Programming

怎樣不產生環

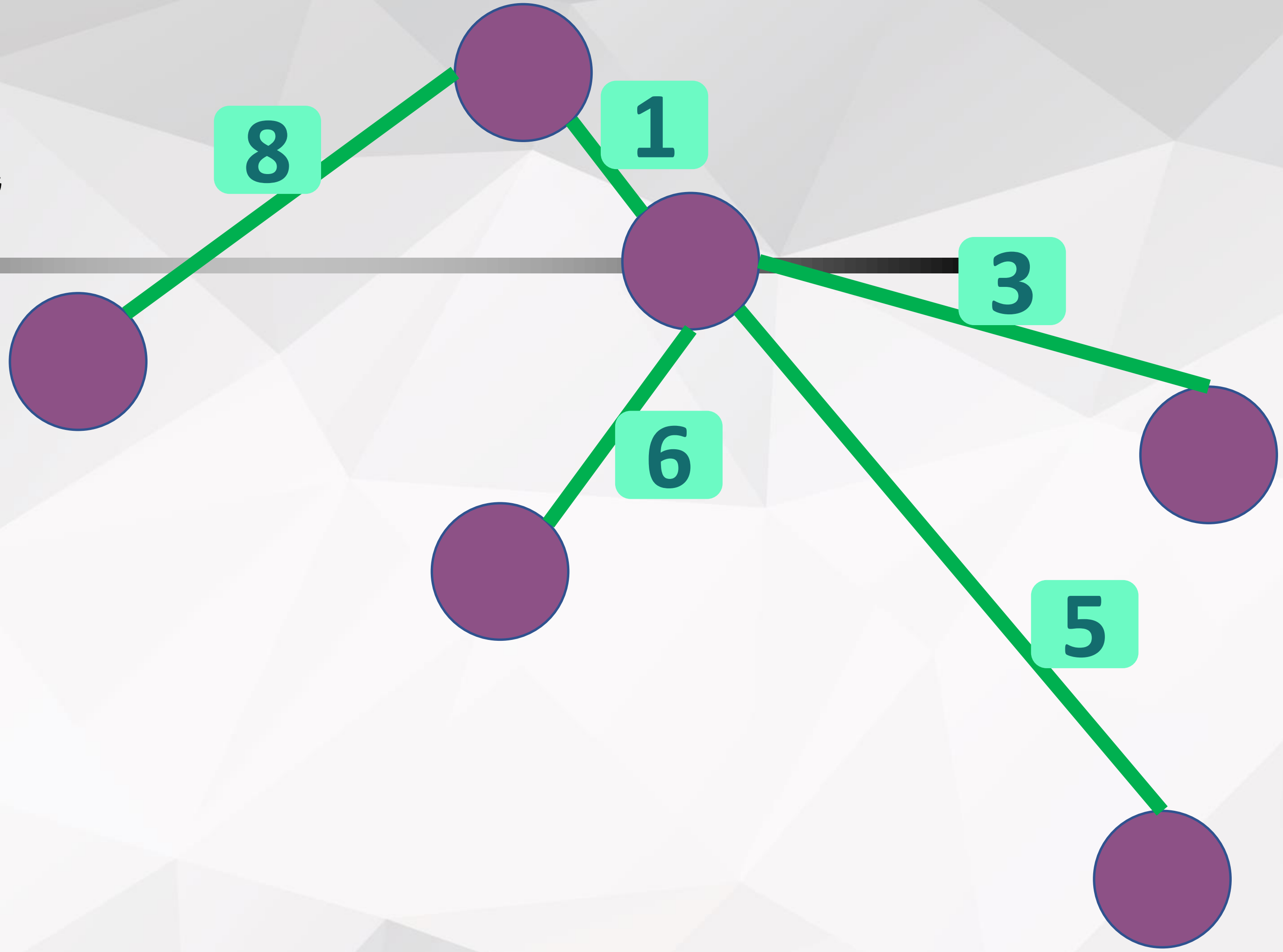
- 在連通塊 A 與連通塊 B 相連時確保不會產生環

$A \neq B \Leftrightarrow$ 加入新的邊不產生環

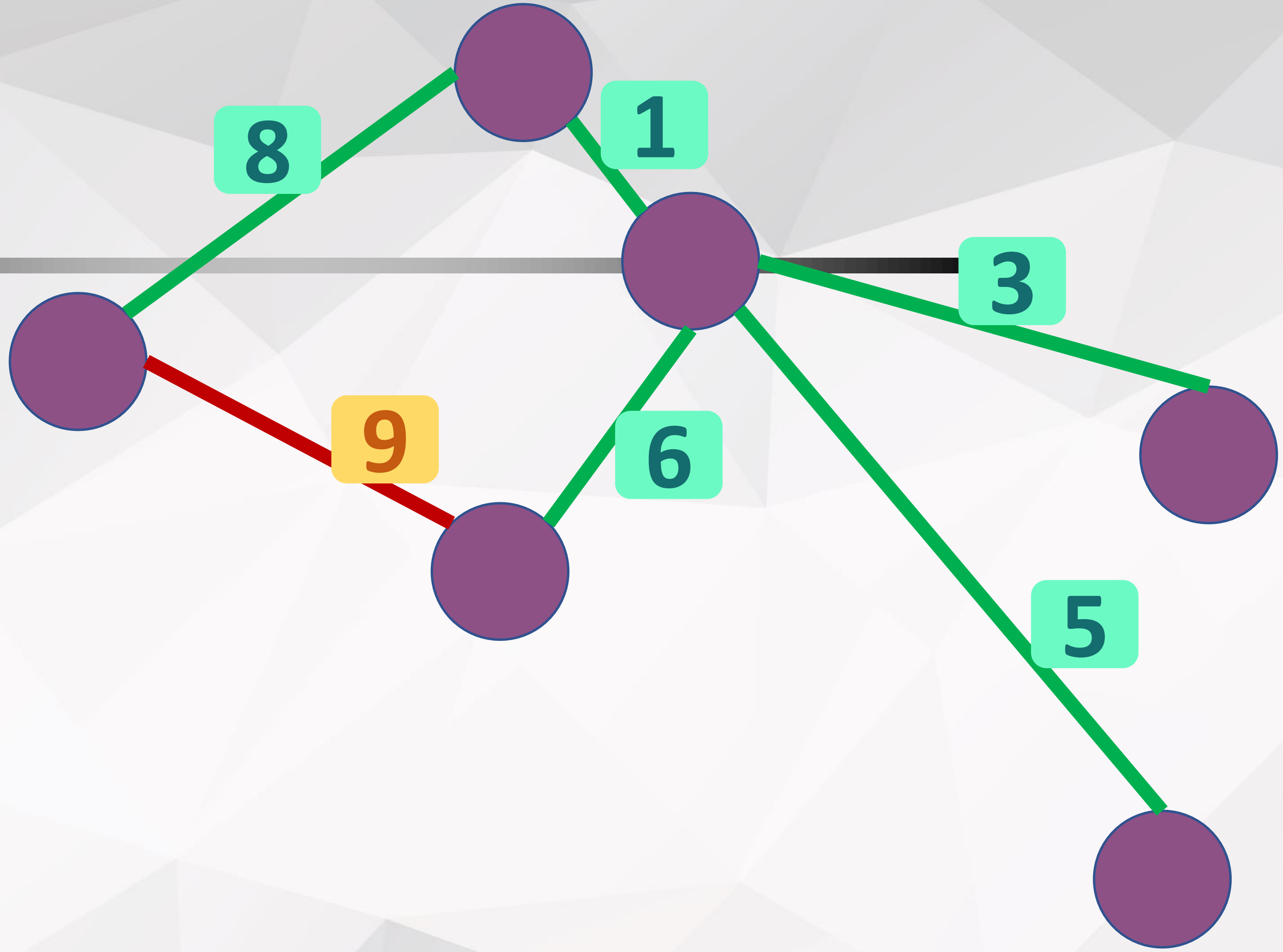
A 連通塊



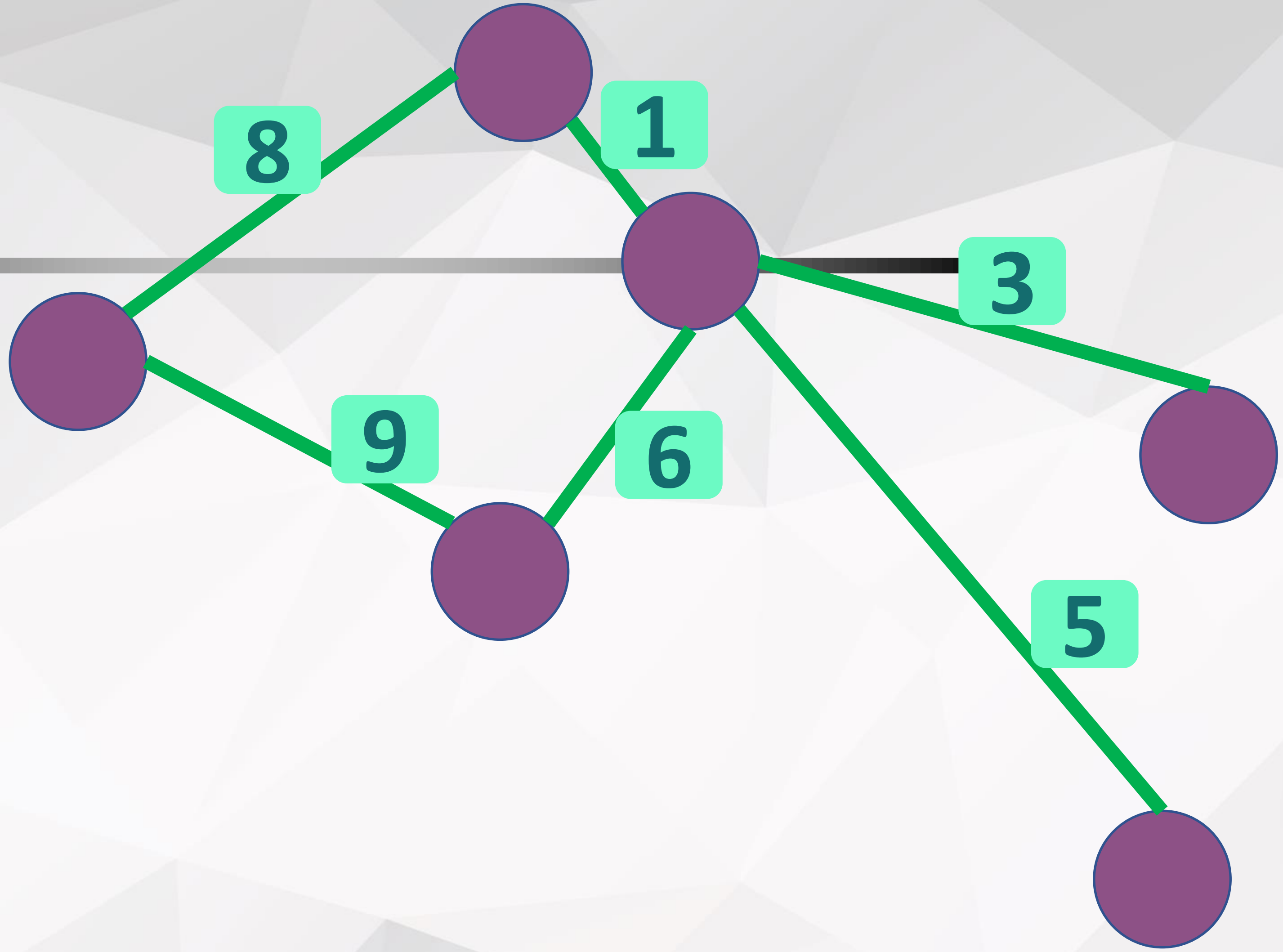
B 連通塊



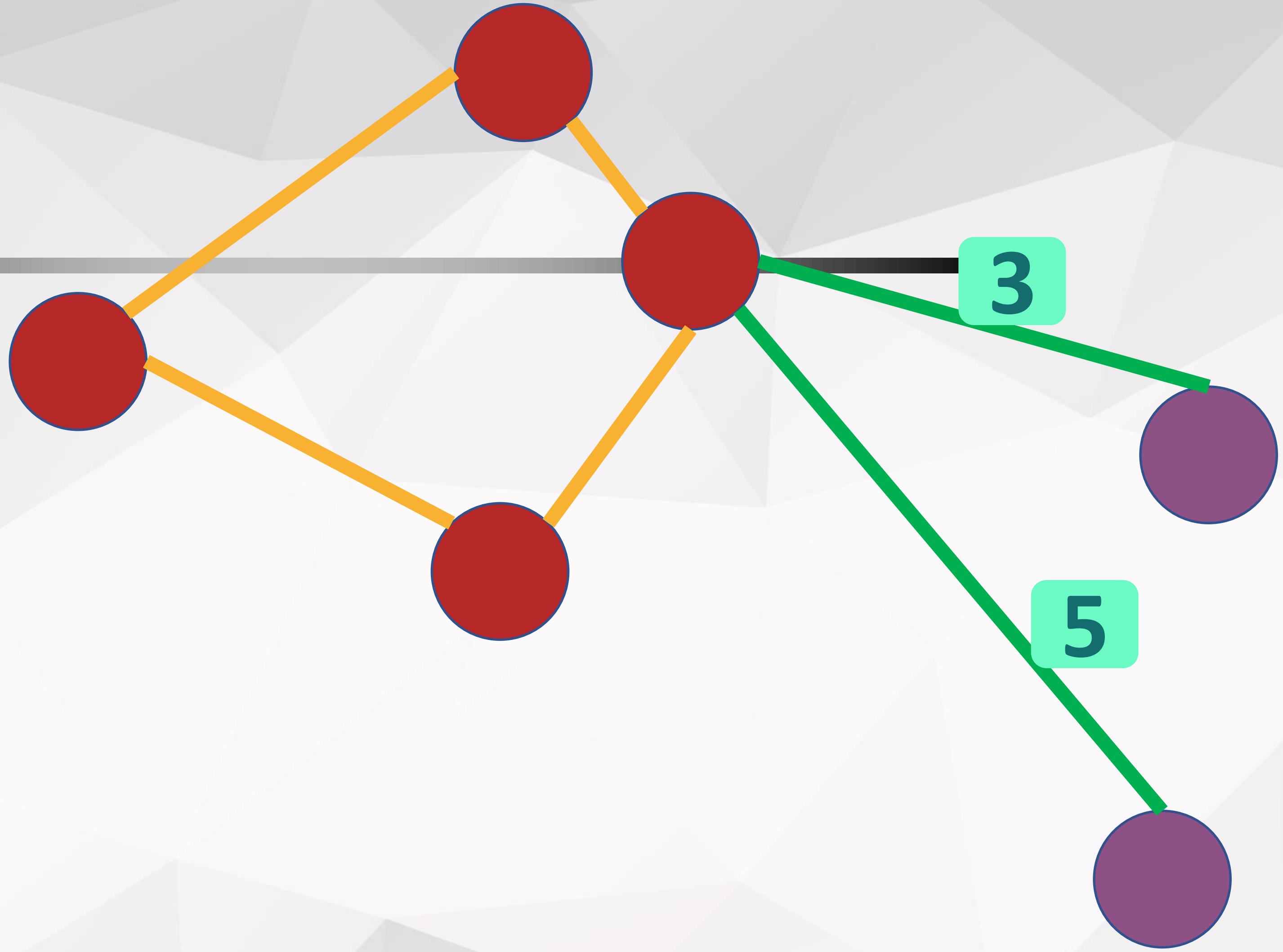
相連



相連



產生環



(⊙д⊙)

怎樣不產生環

- 在連通塊 A 與連通塊 B 相連時確保不會產生環

$$A \neq B \Leftrightarrow \text{加入新的邊不產生環}$$

怎樣不產生環

- 在連通塊 A 與連通塊 B 相連時確保不會產生環

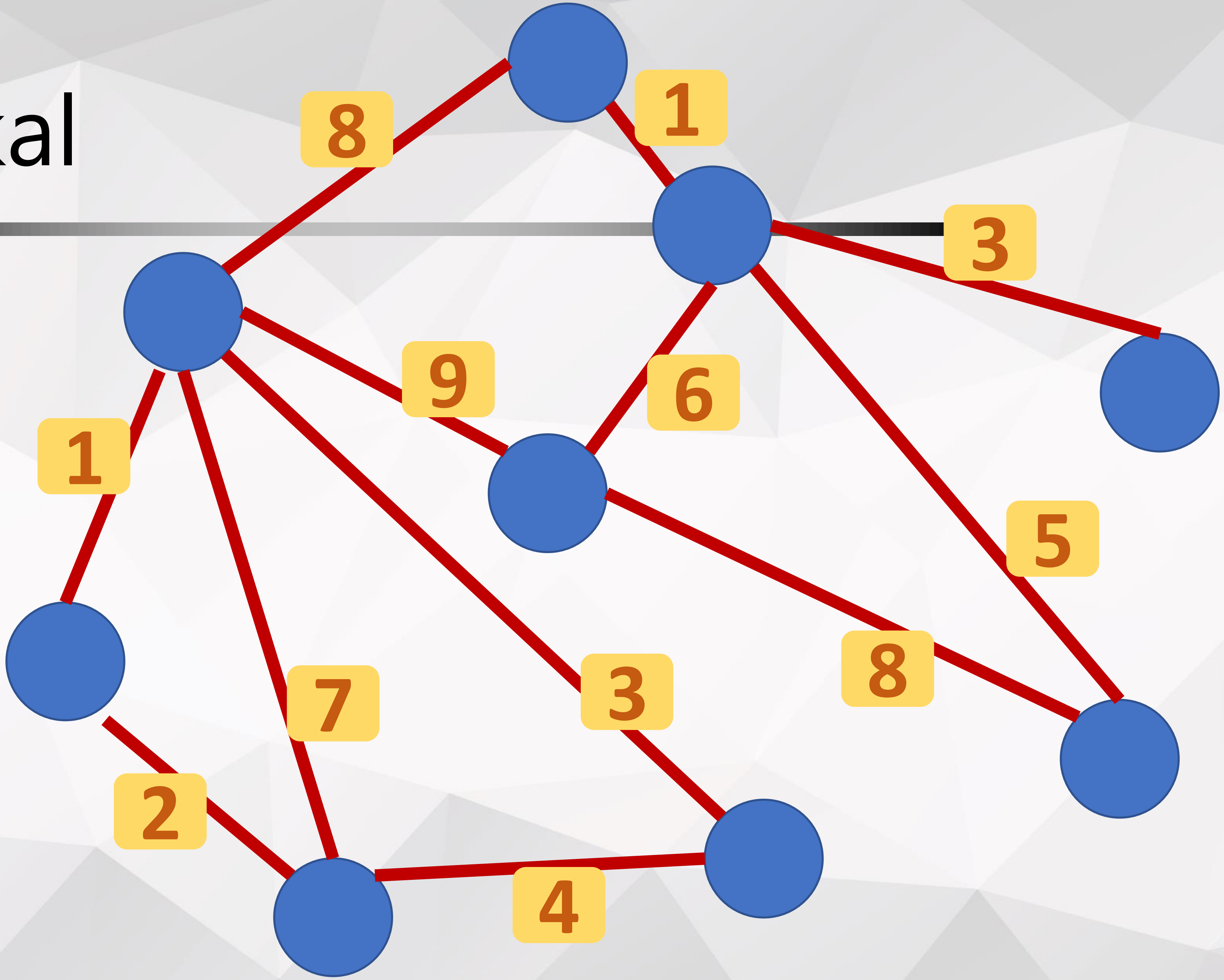
$A \neq B \Leftrightarrow$ 加入新的邊不產生環

- 所以過程中要確保挑的**連通塊互為獨立**

Kruskal 演算法

- 直覺的，每次相連選一個**合法**且**權重最小**的邊
- 最終得到的生成樹，就是最小生成樹

Kruskal



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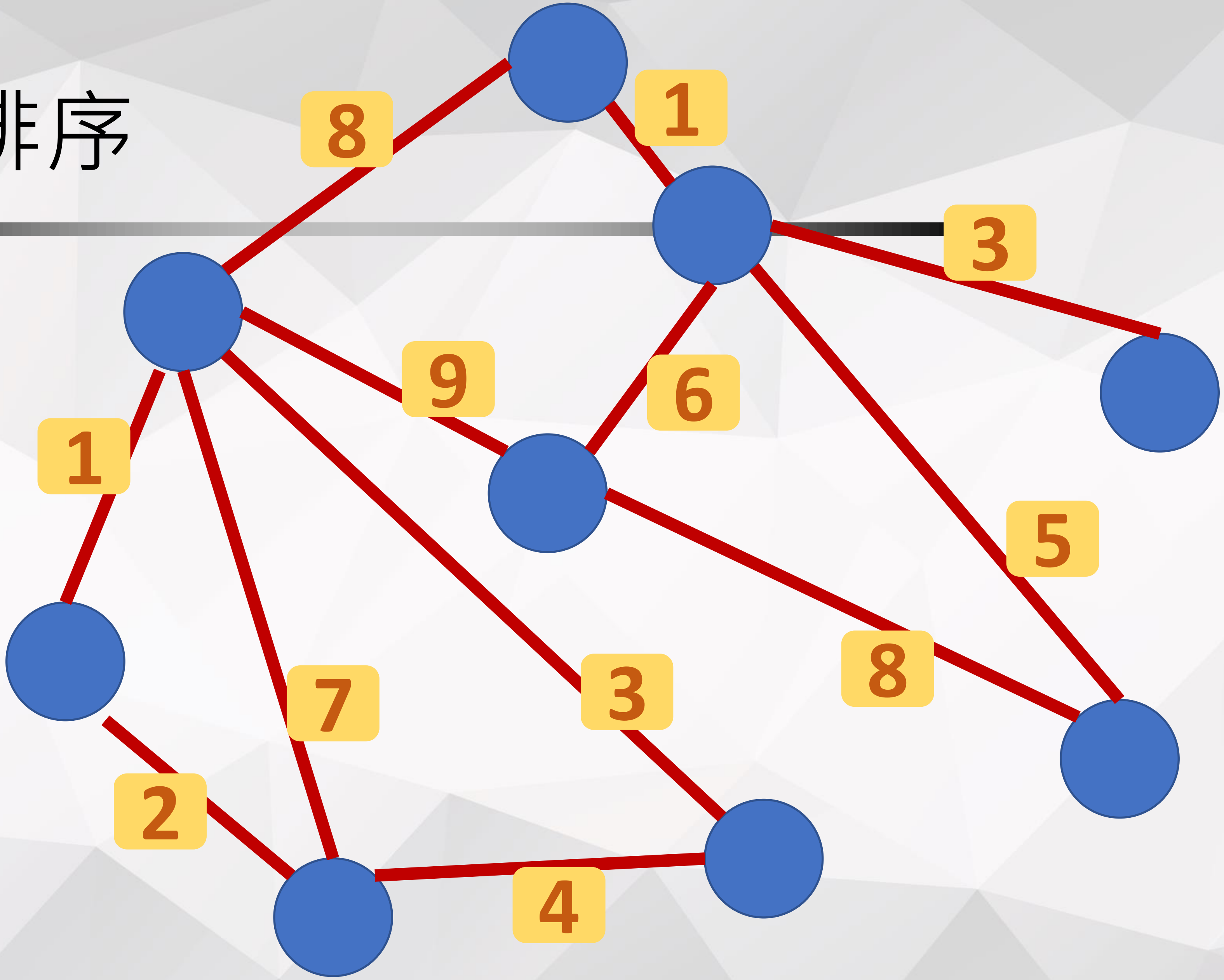
權重排序

1
1
2
3
3
4
5
6
7
8
8
9



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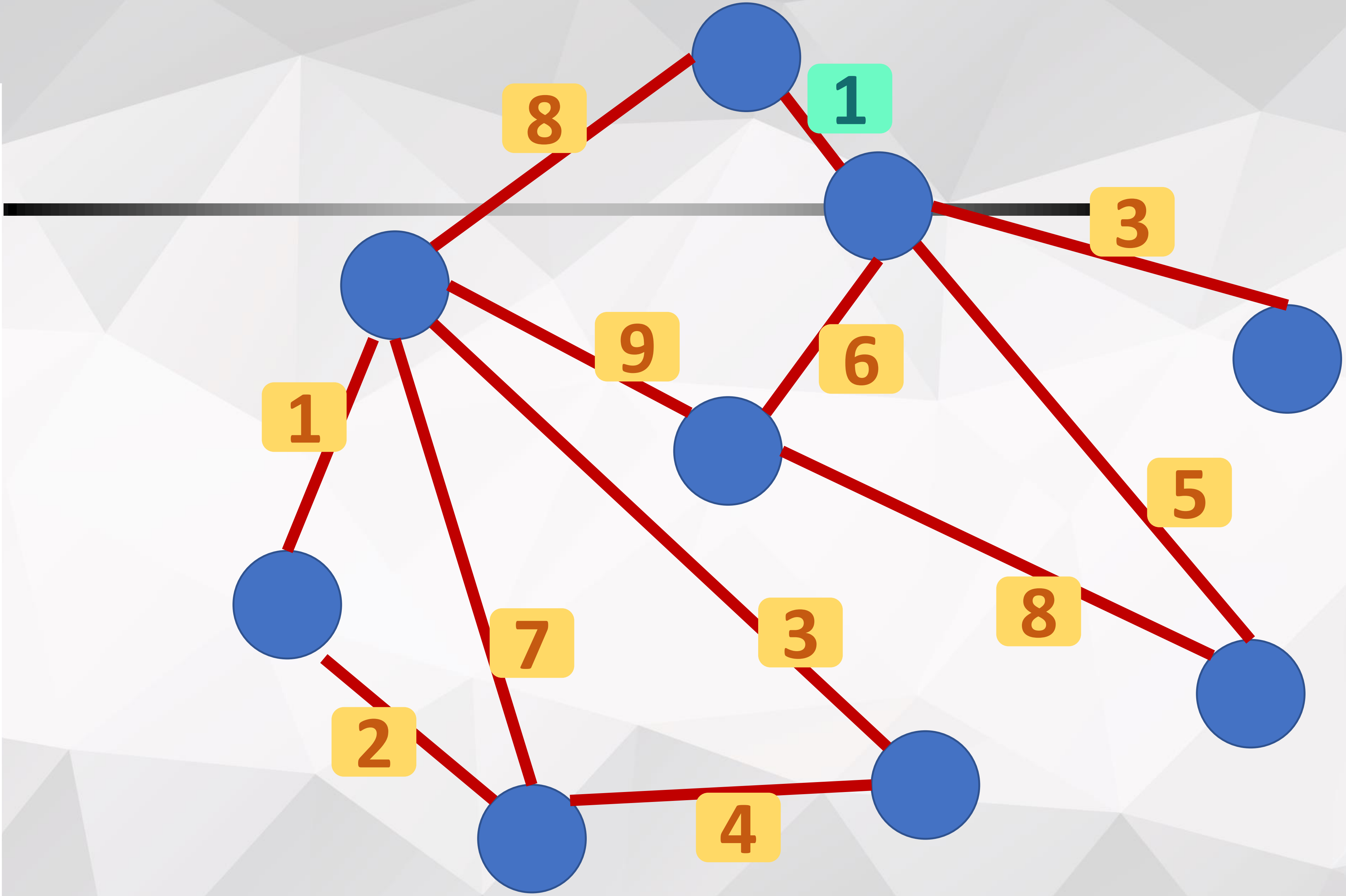

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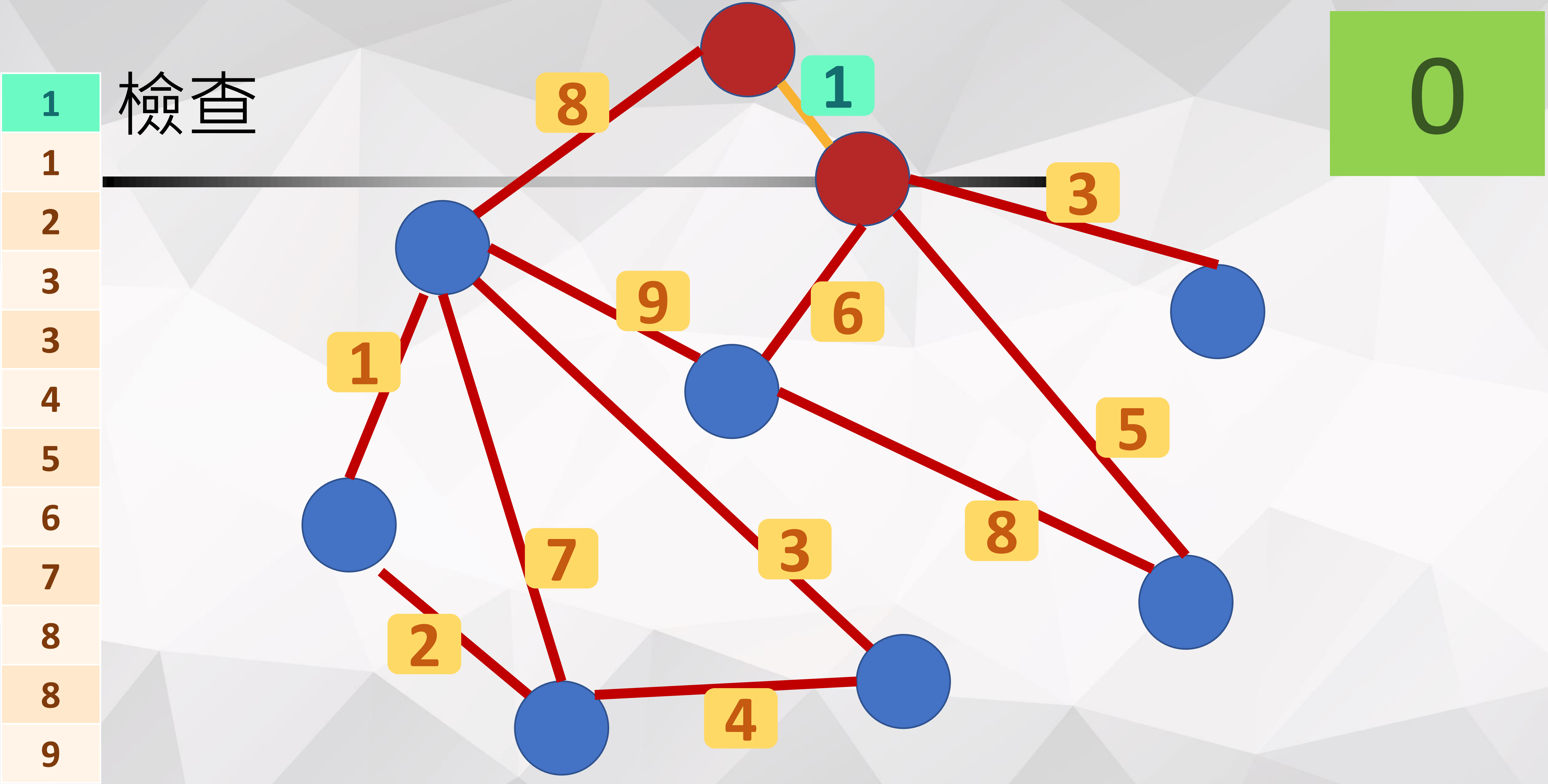
Competitive Programming

檢查



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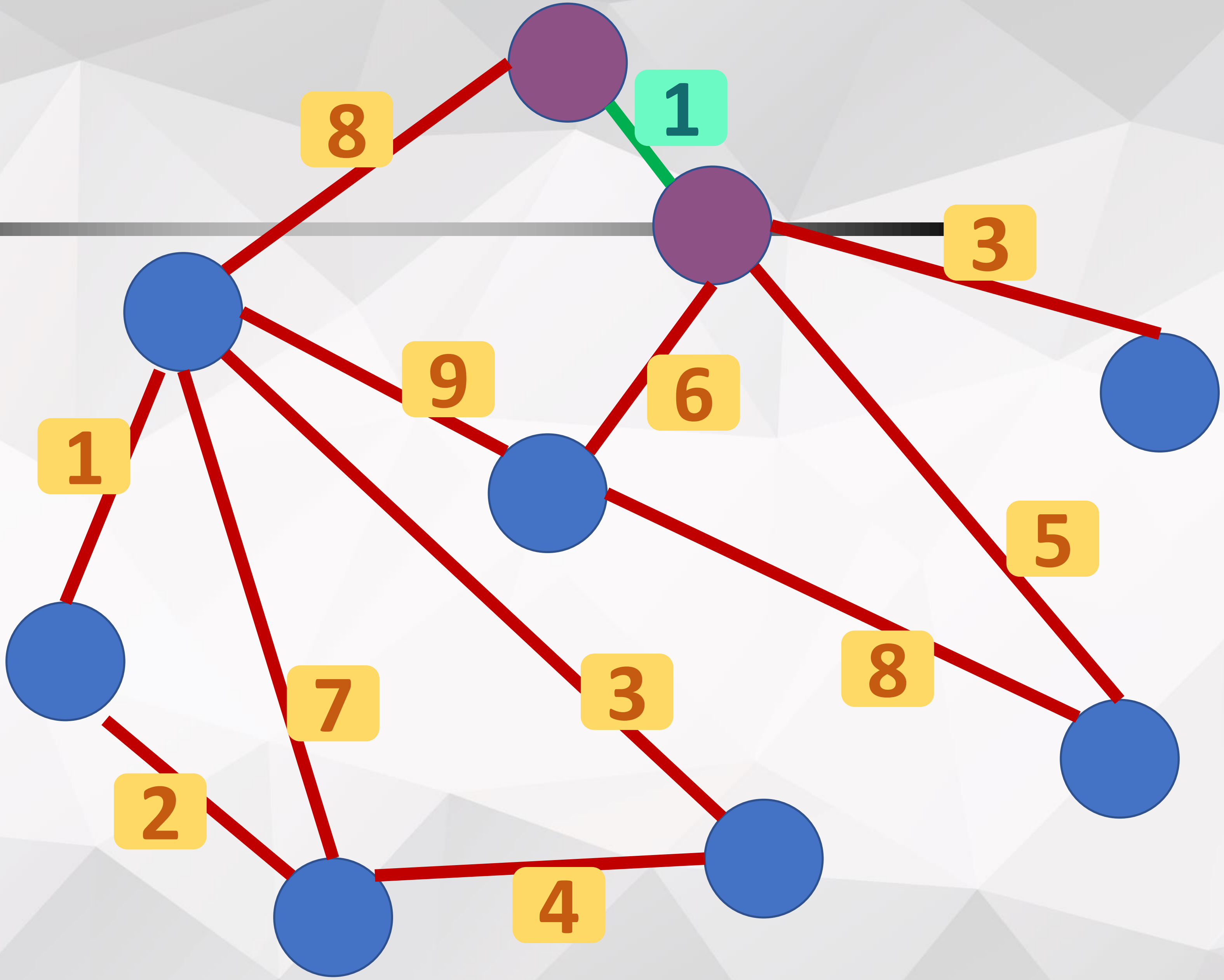
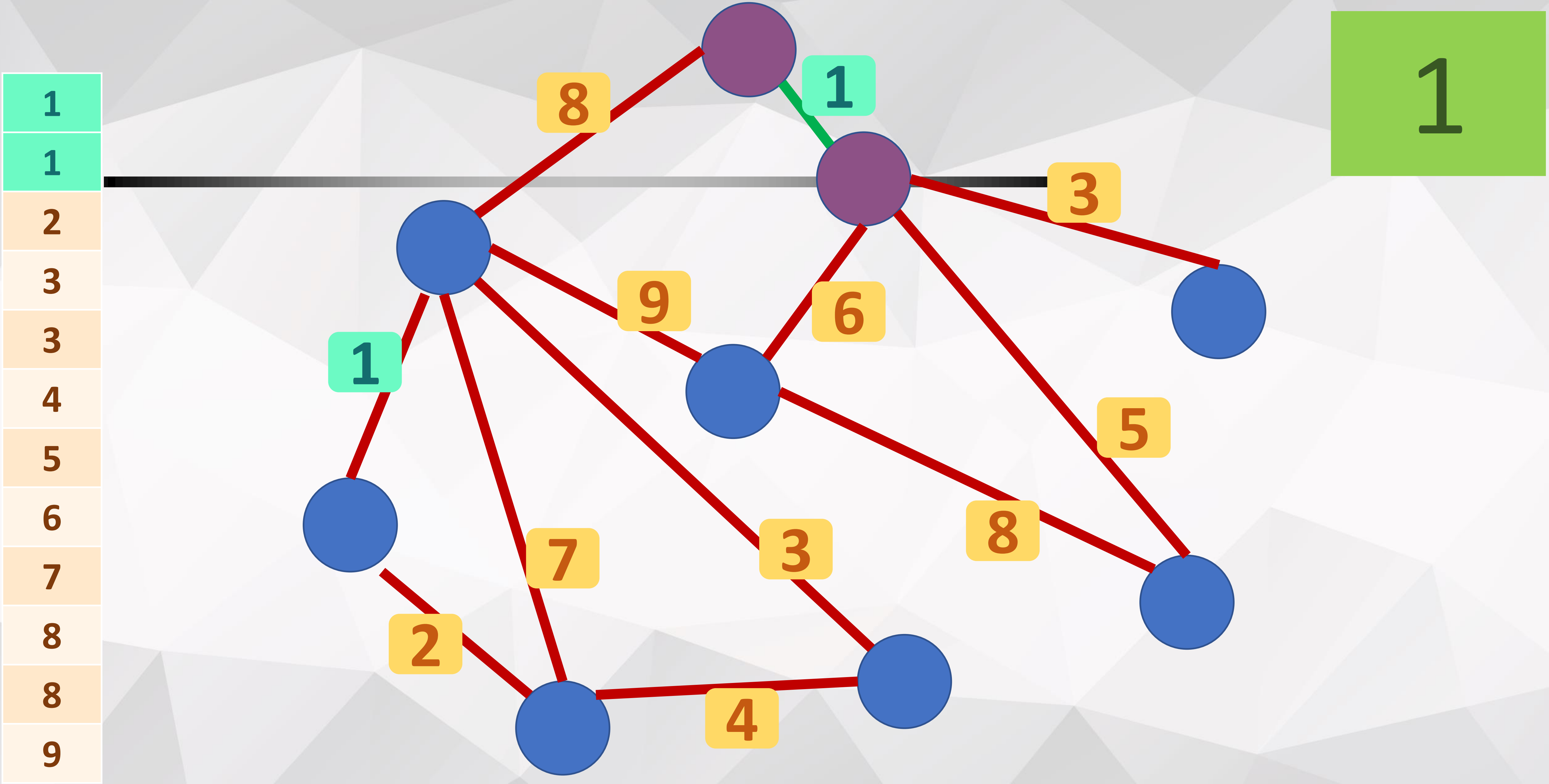


Competitive Programming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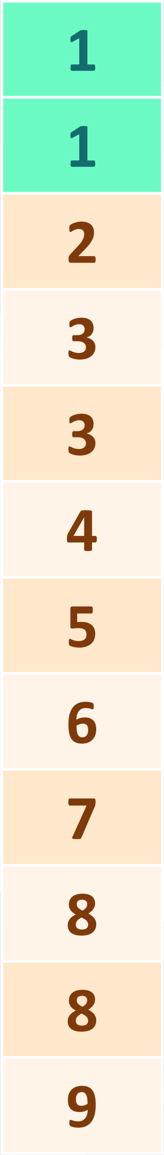
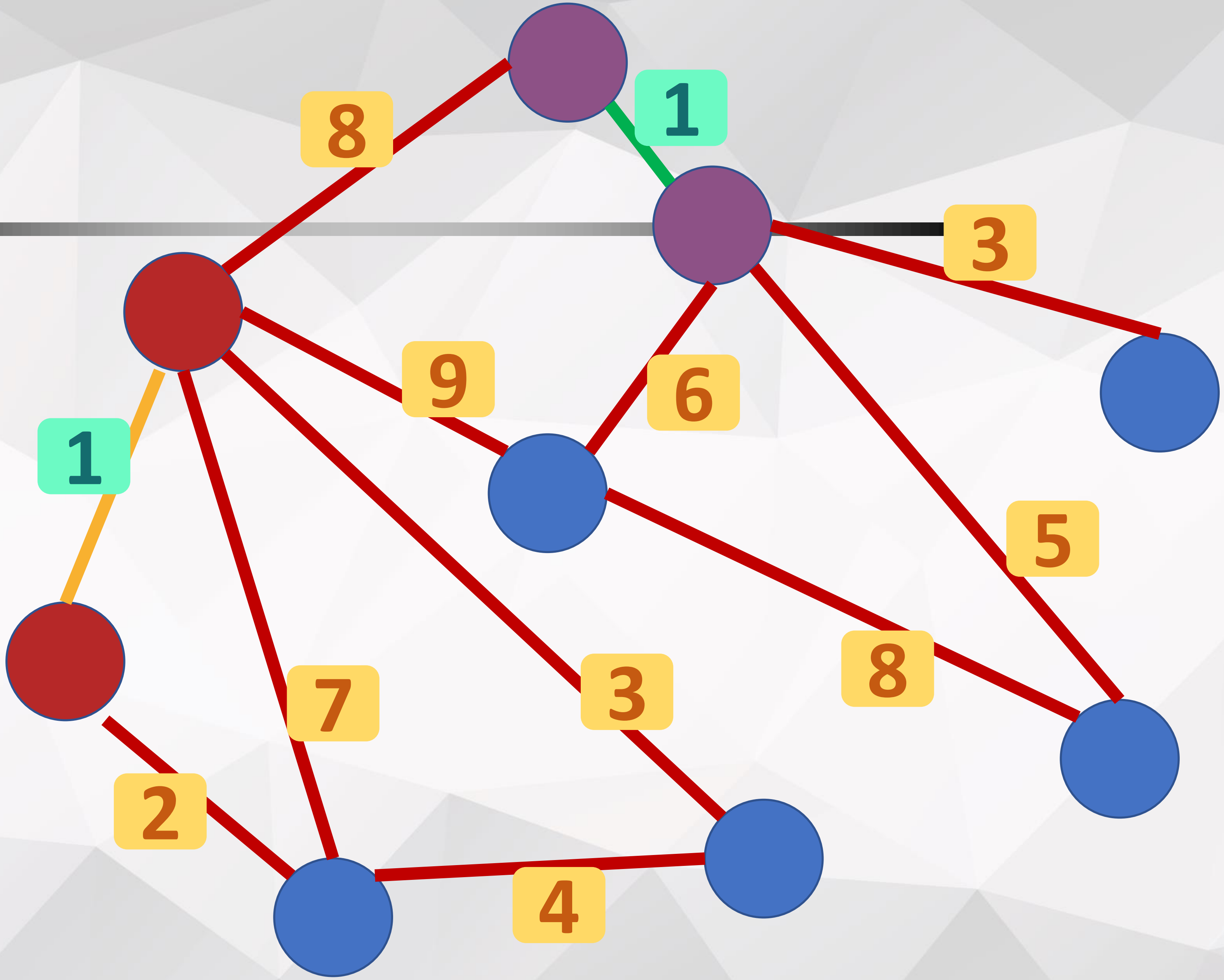
檢查





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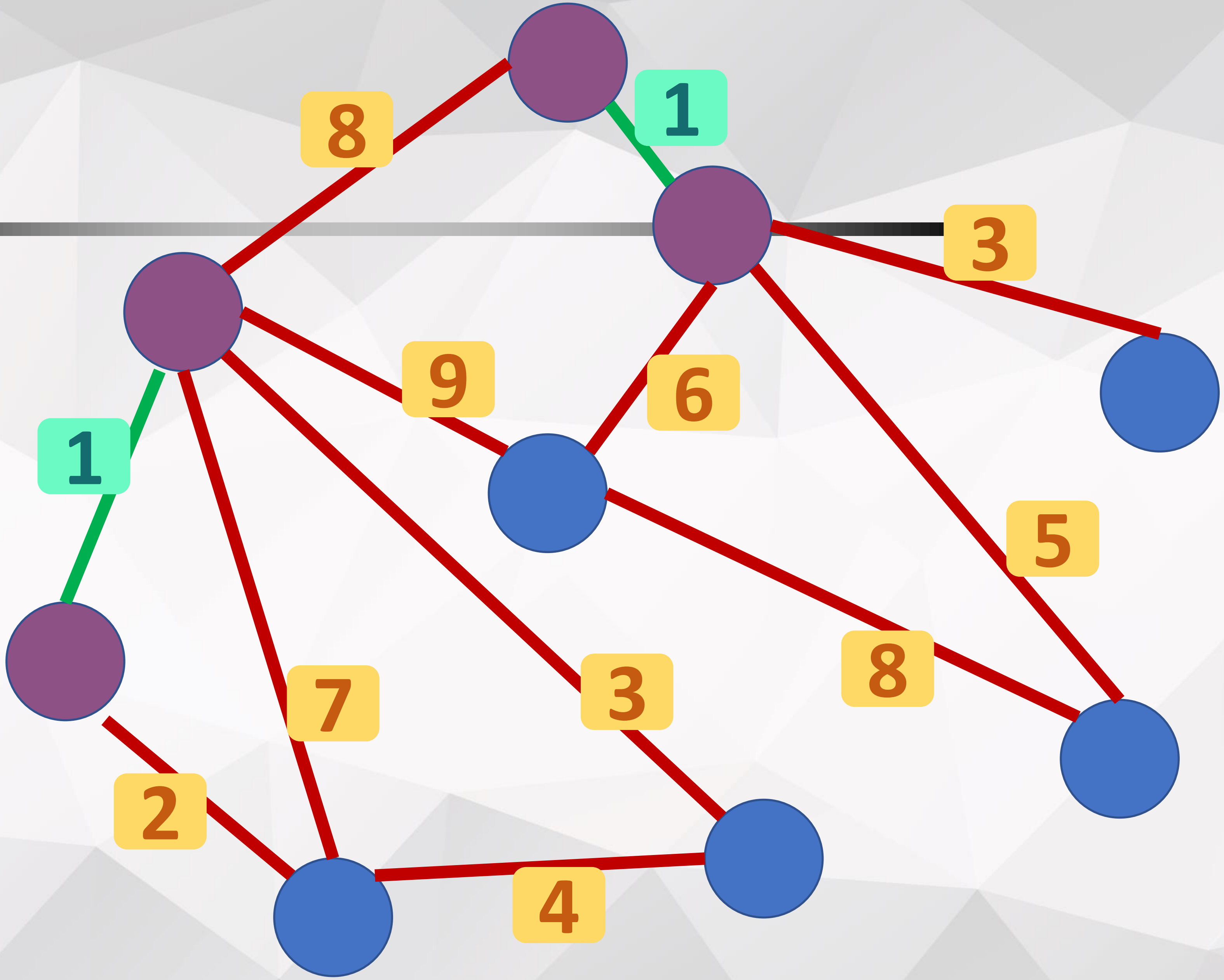

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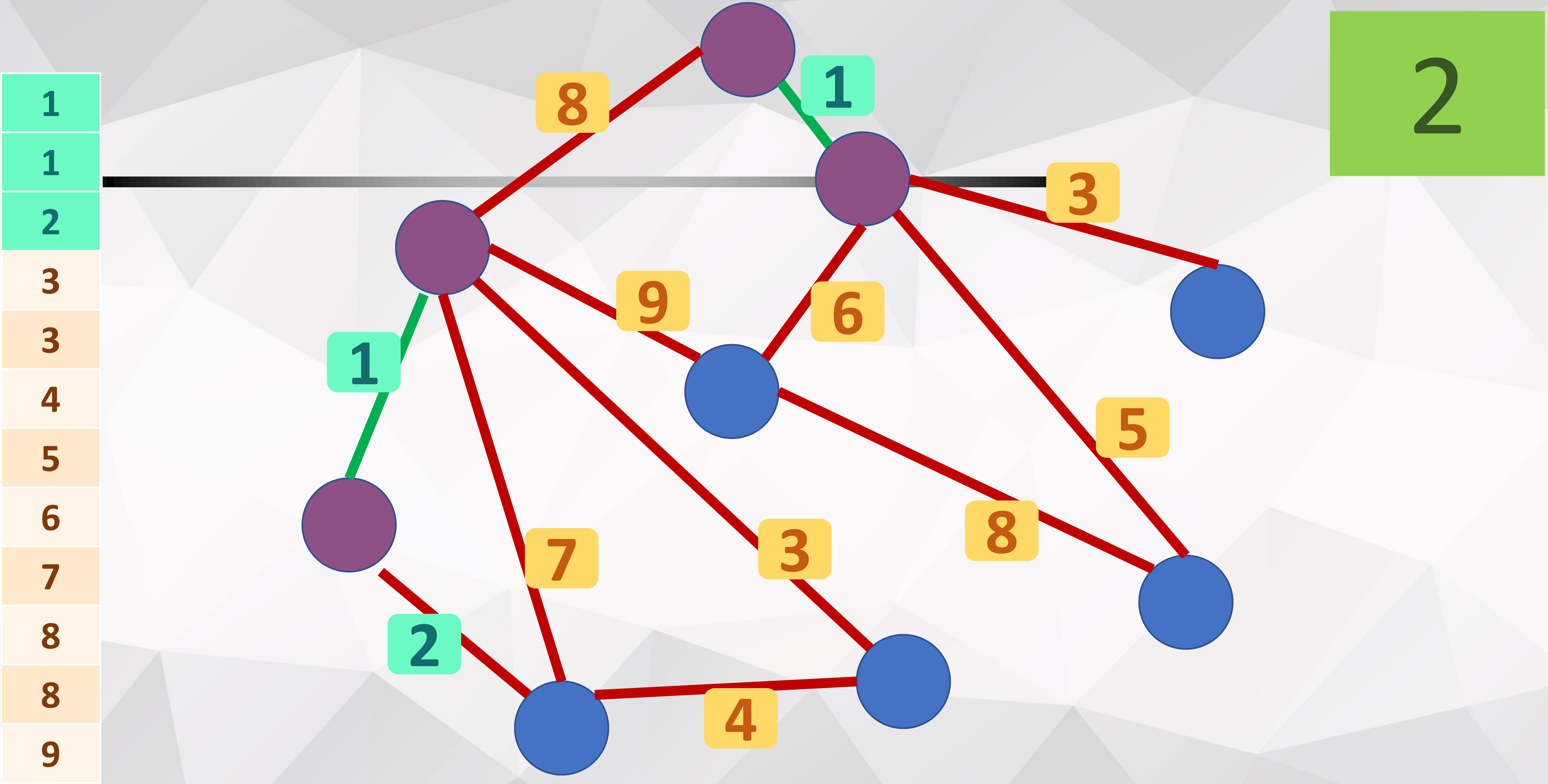
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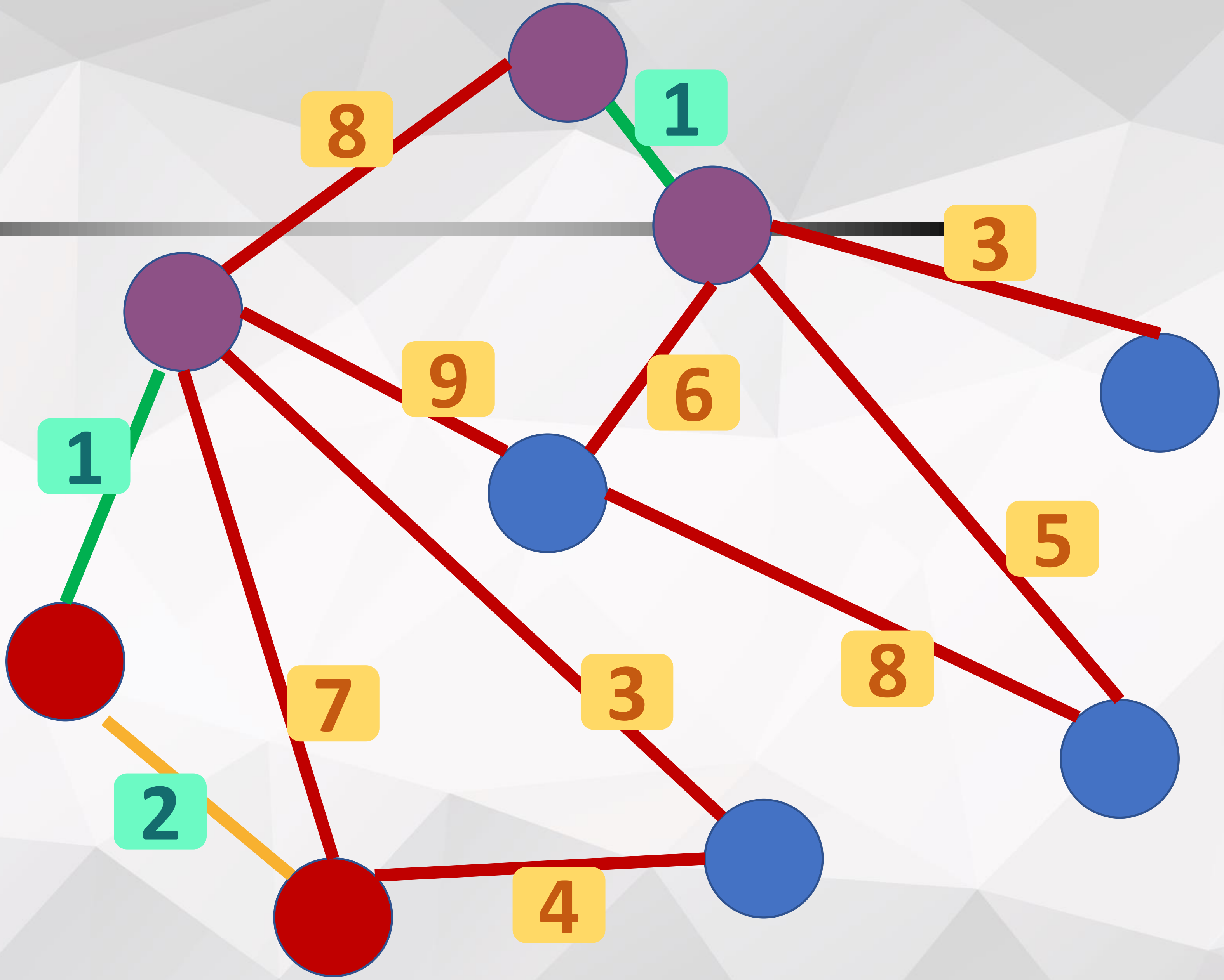

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Competitive Programming

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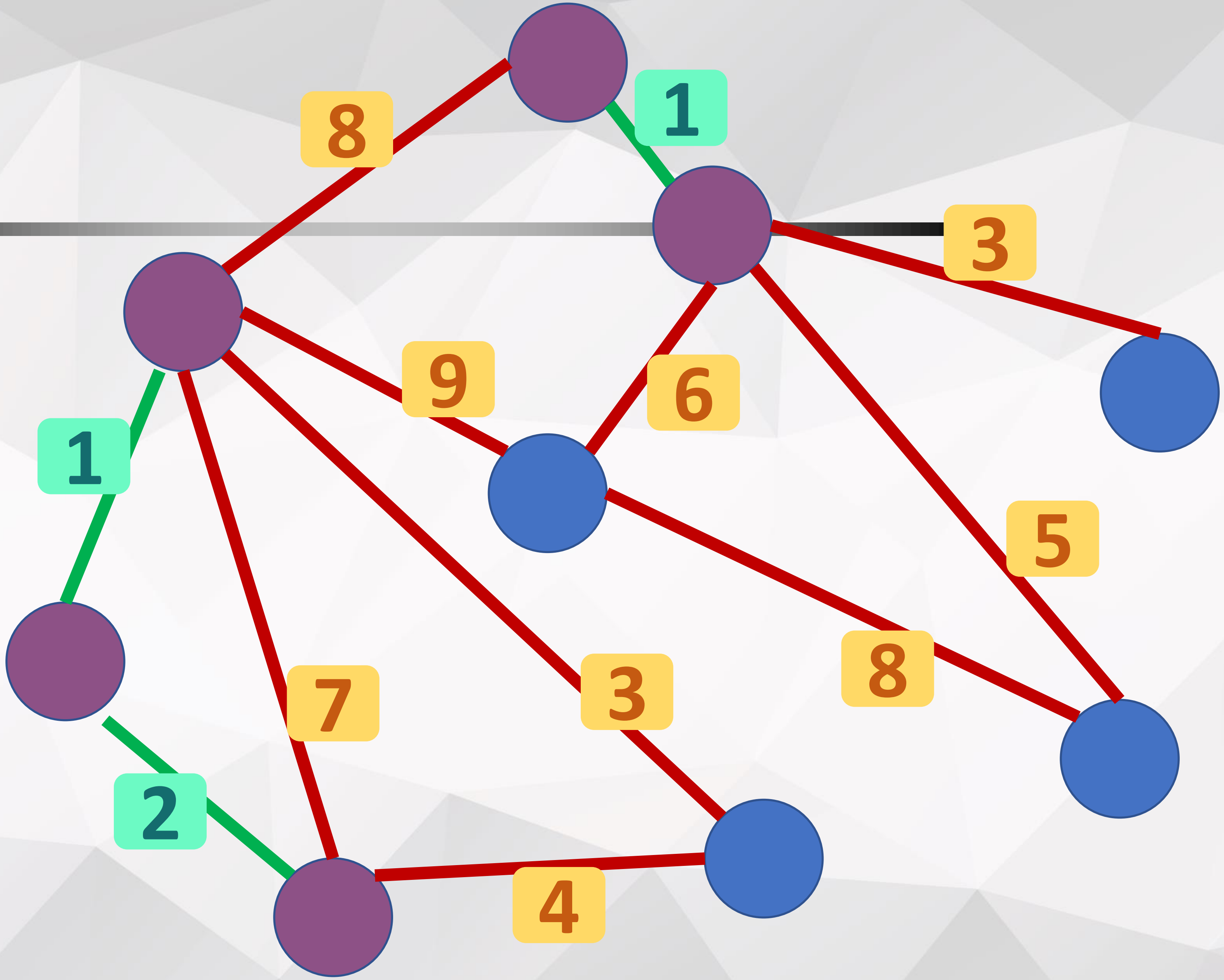

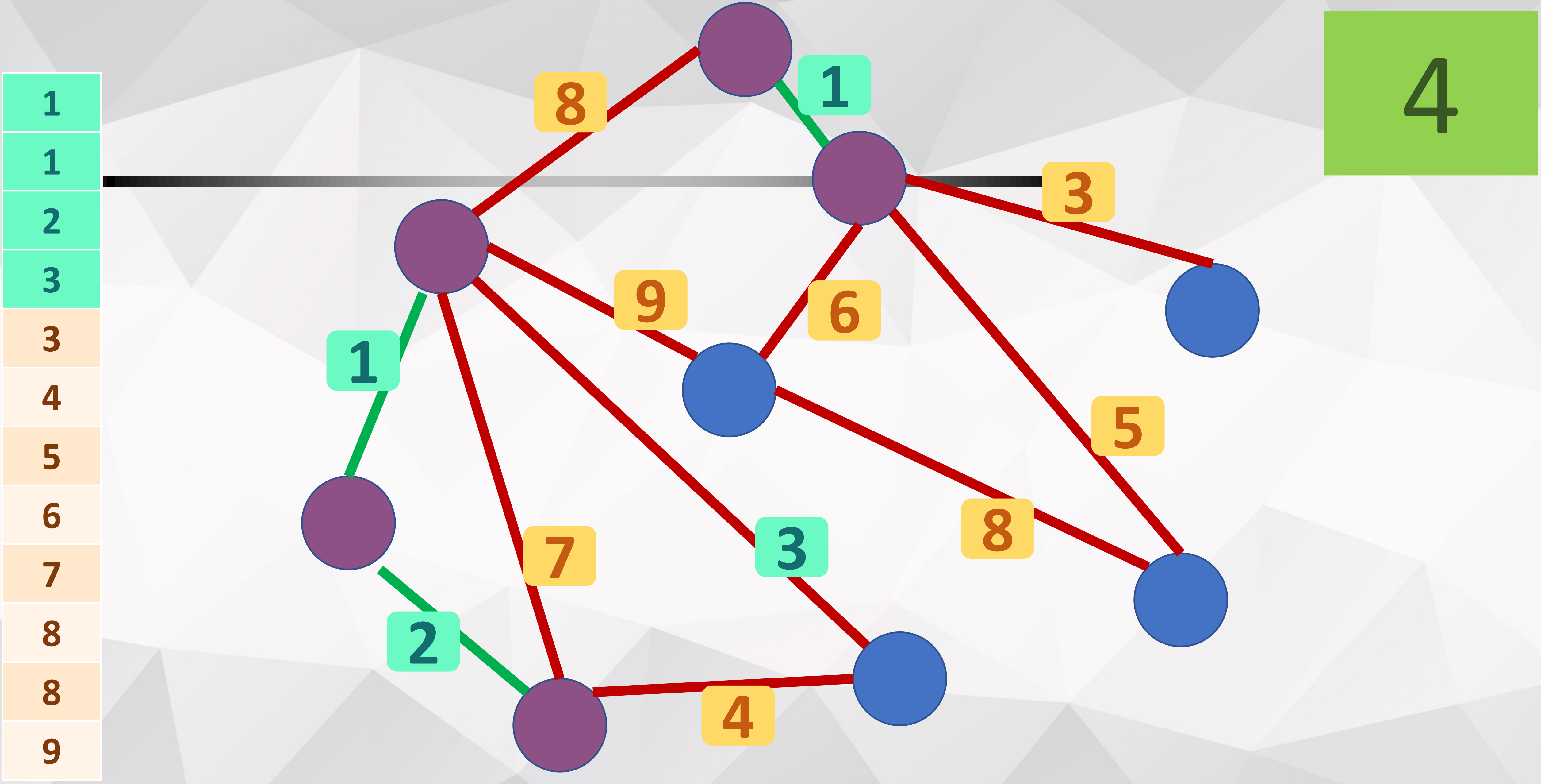

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檢查

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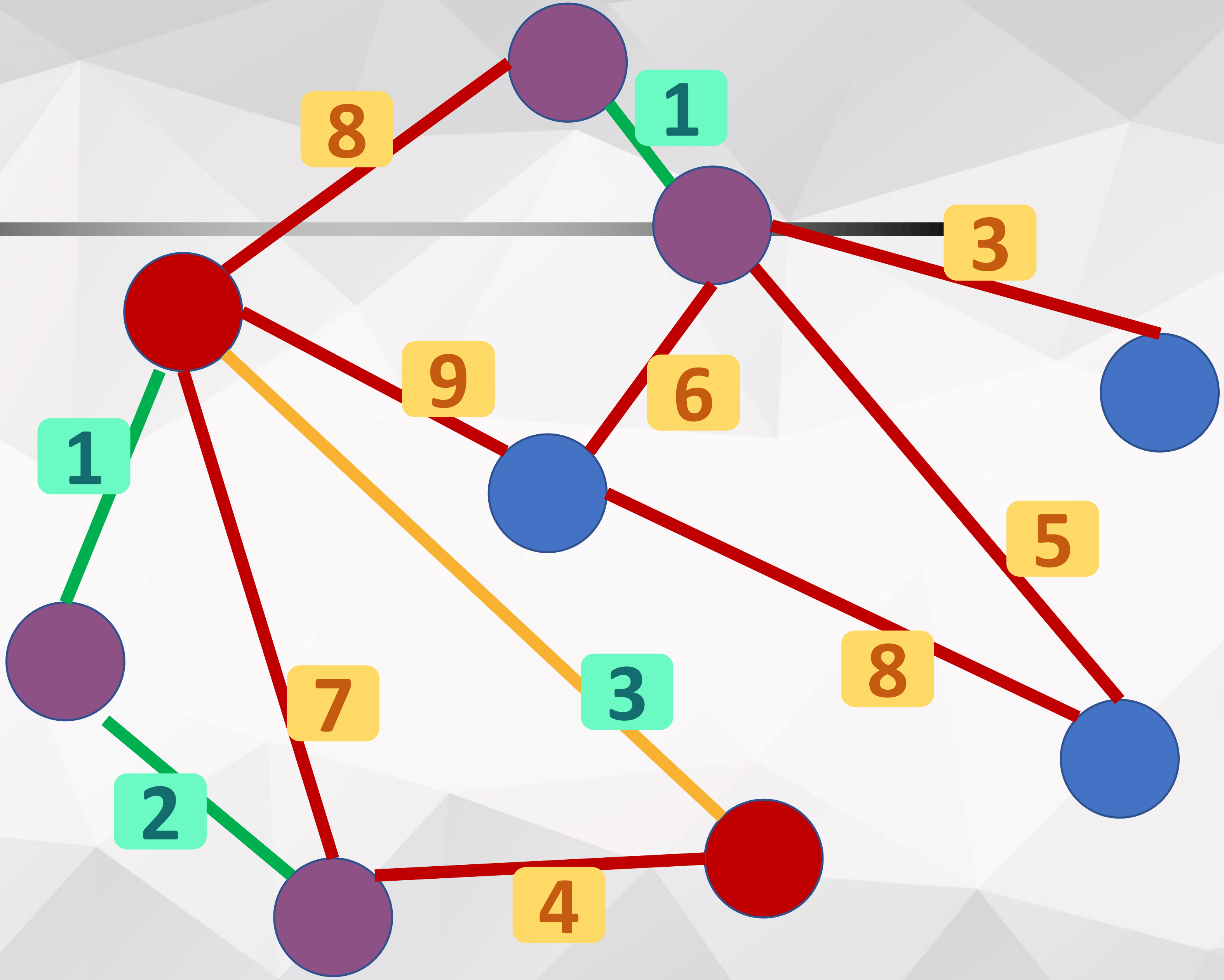

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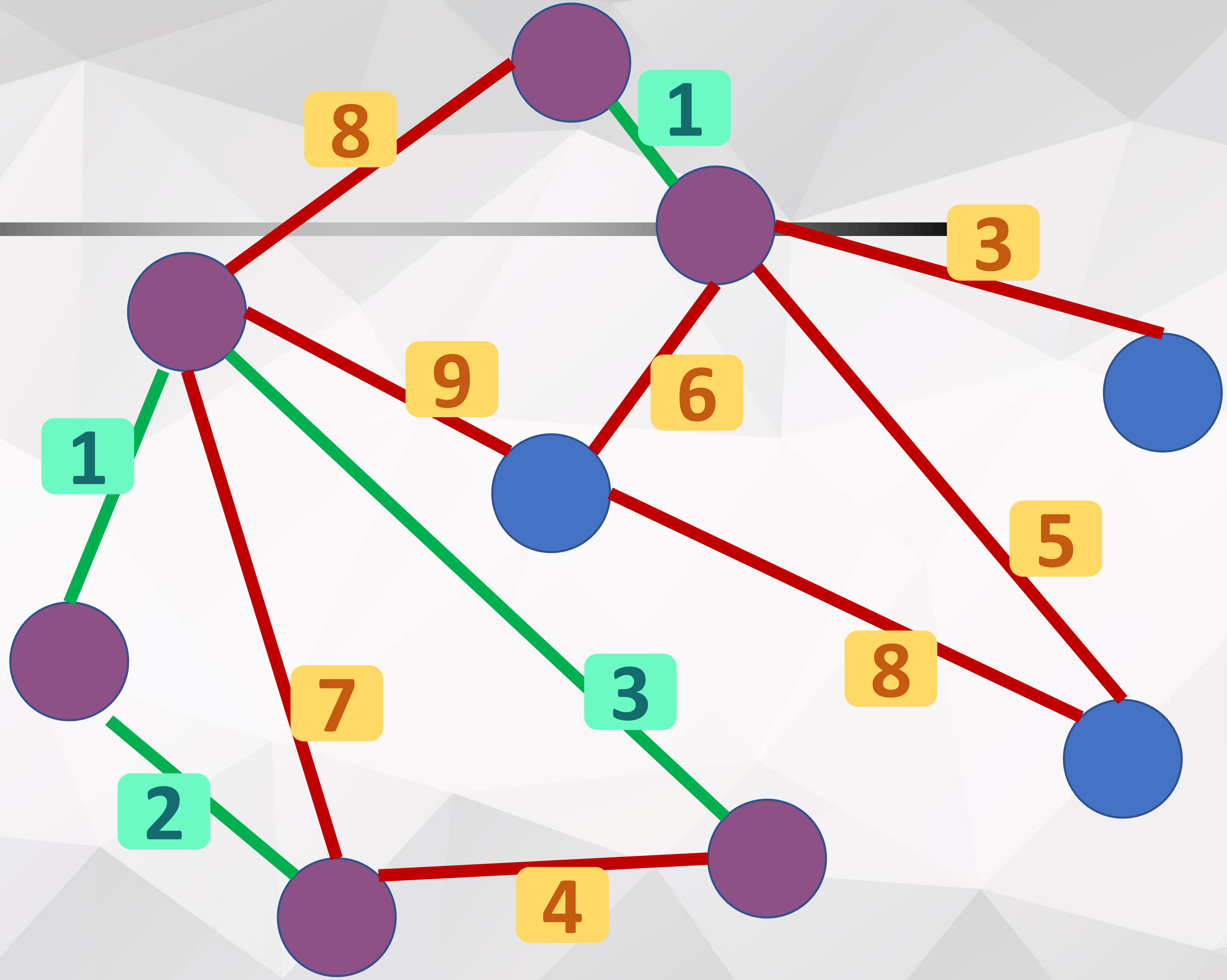

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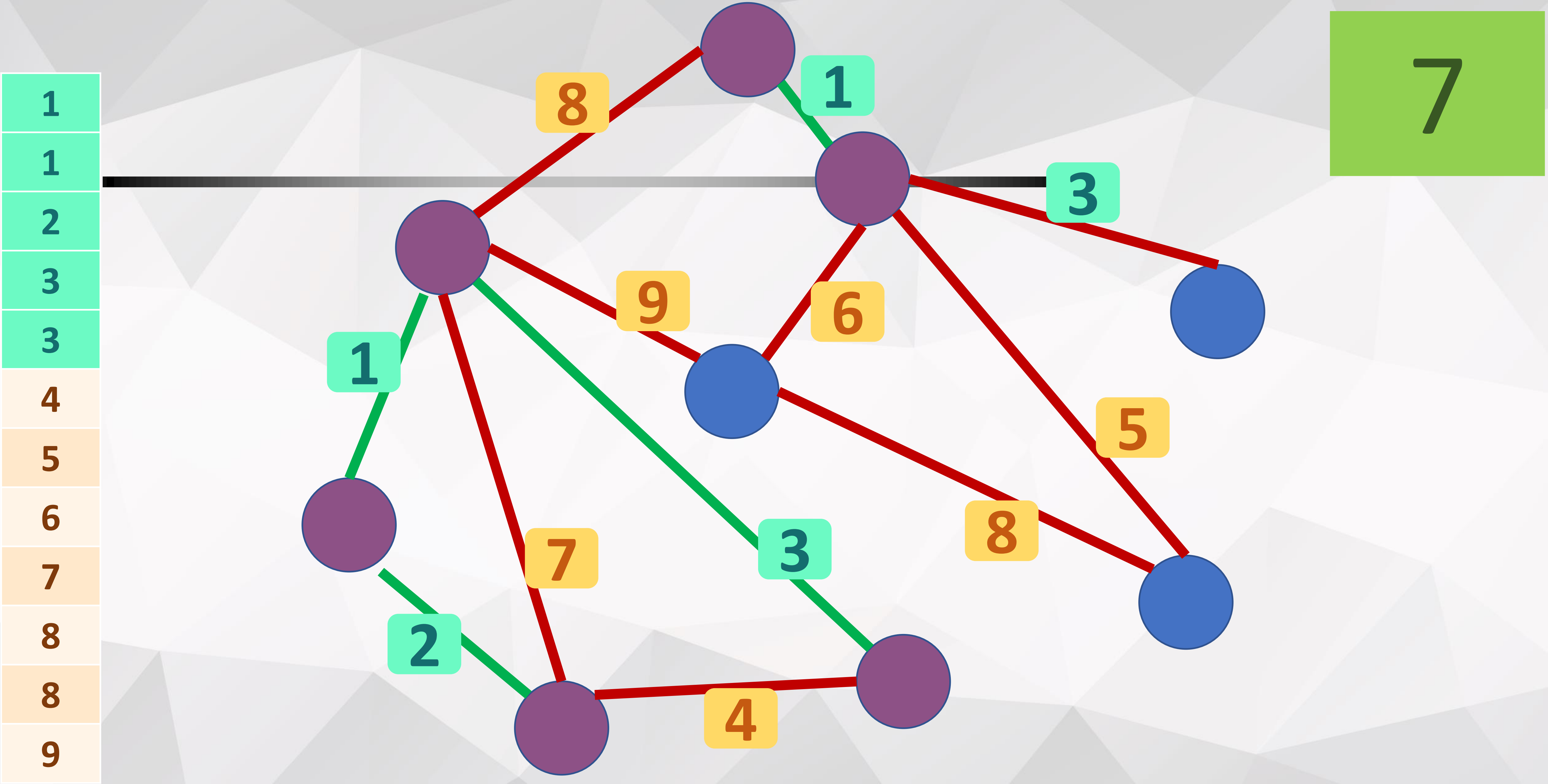
Competitive Programming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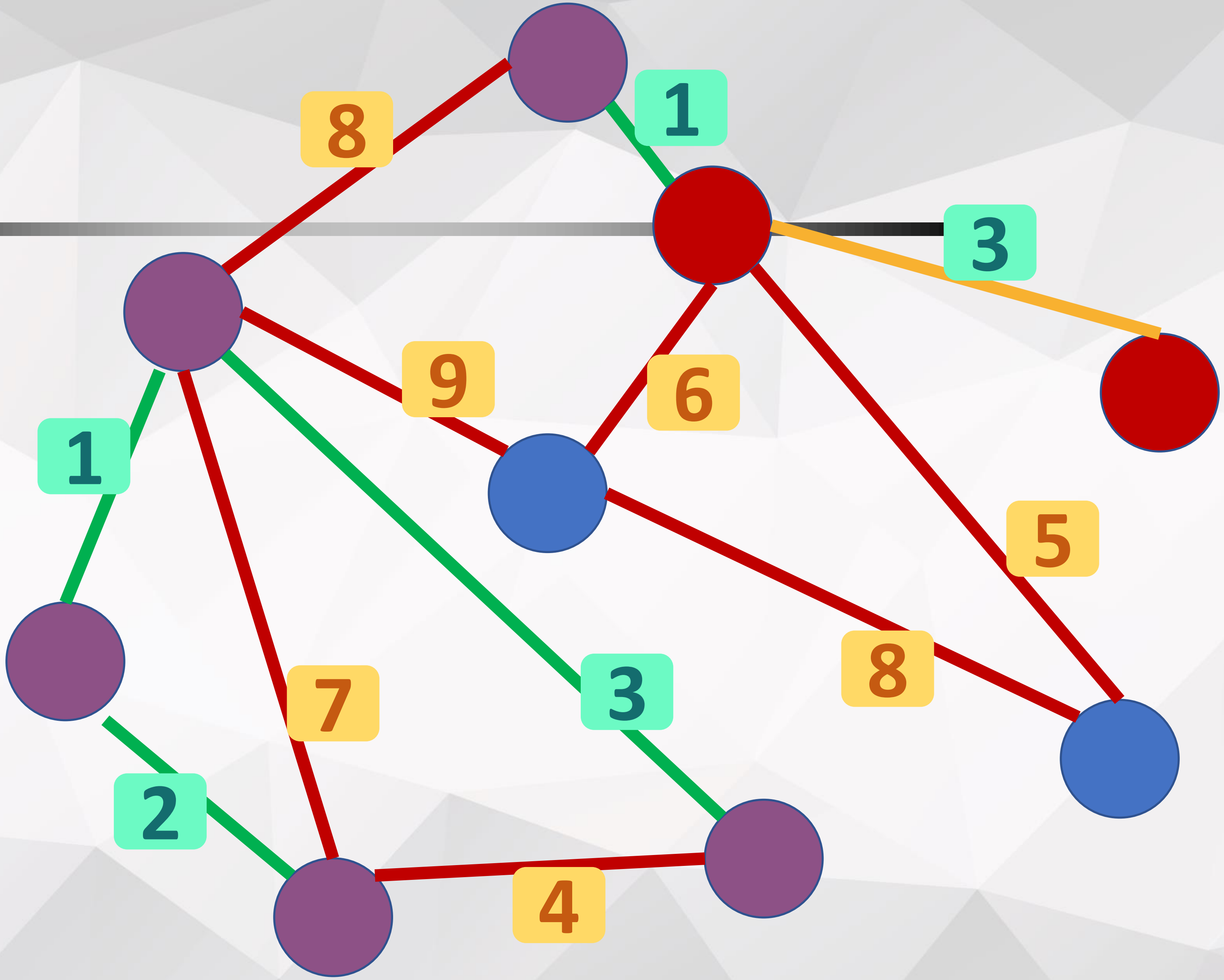

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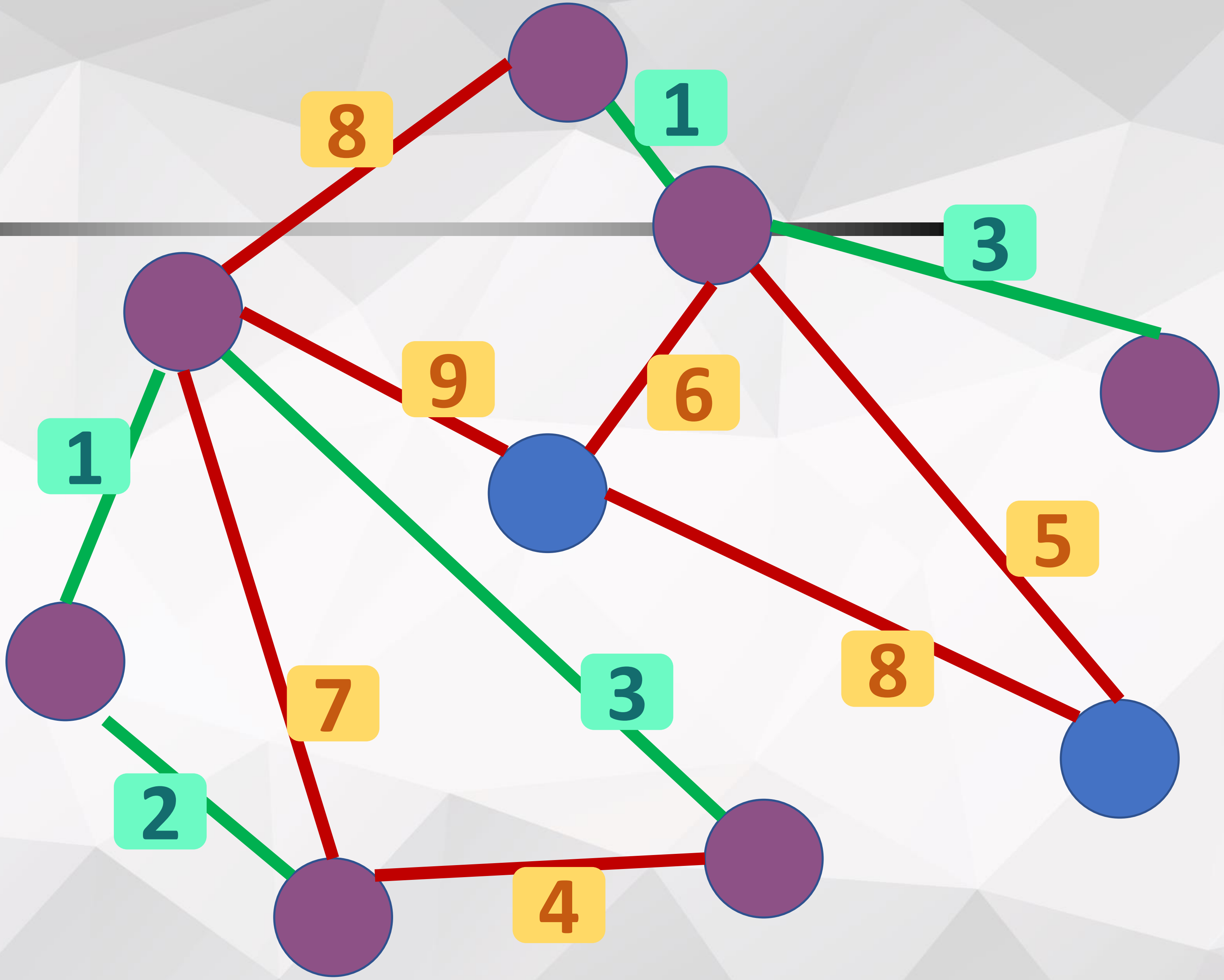

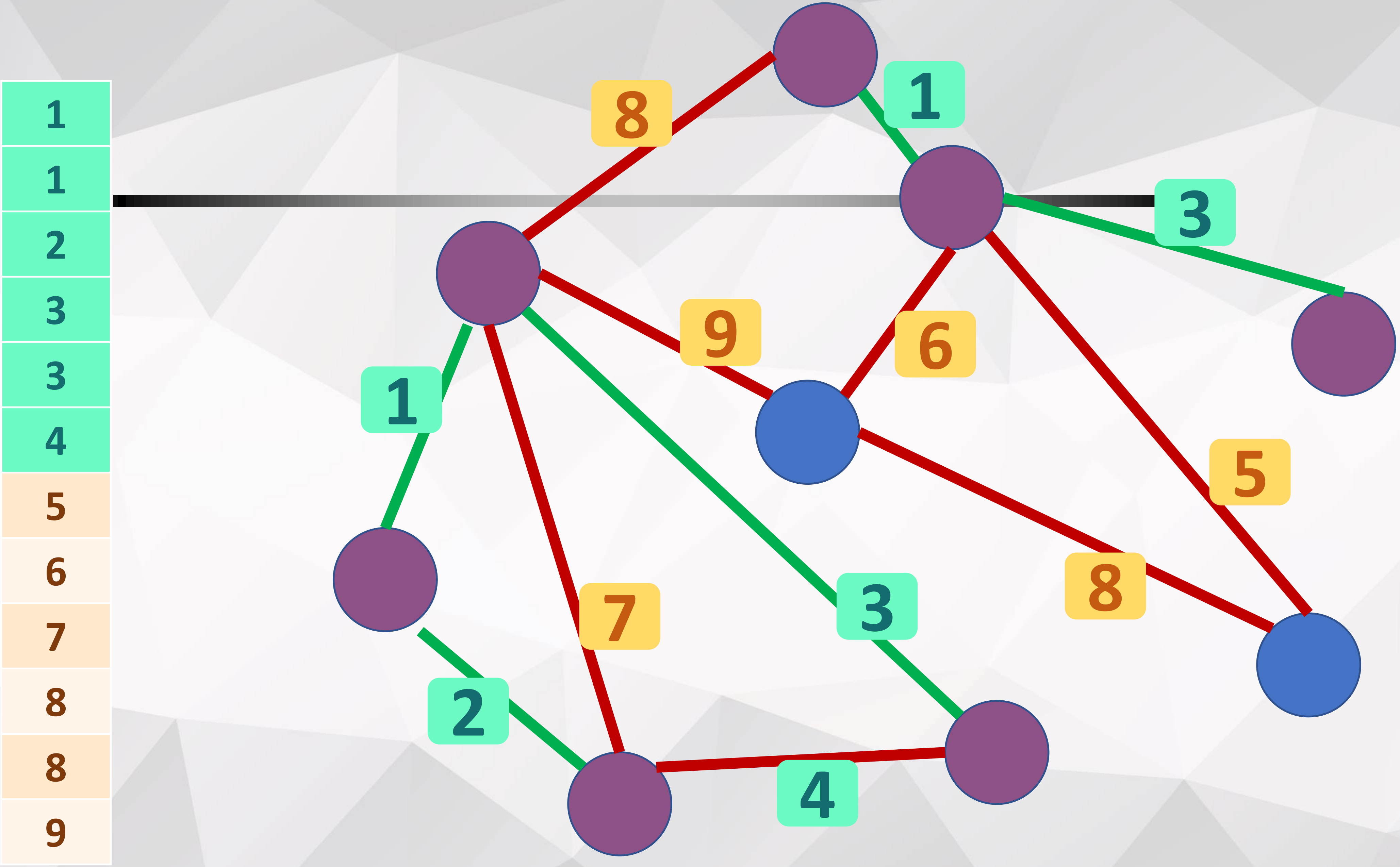

10

Competitive Programming

檢查

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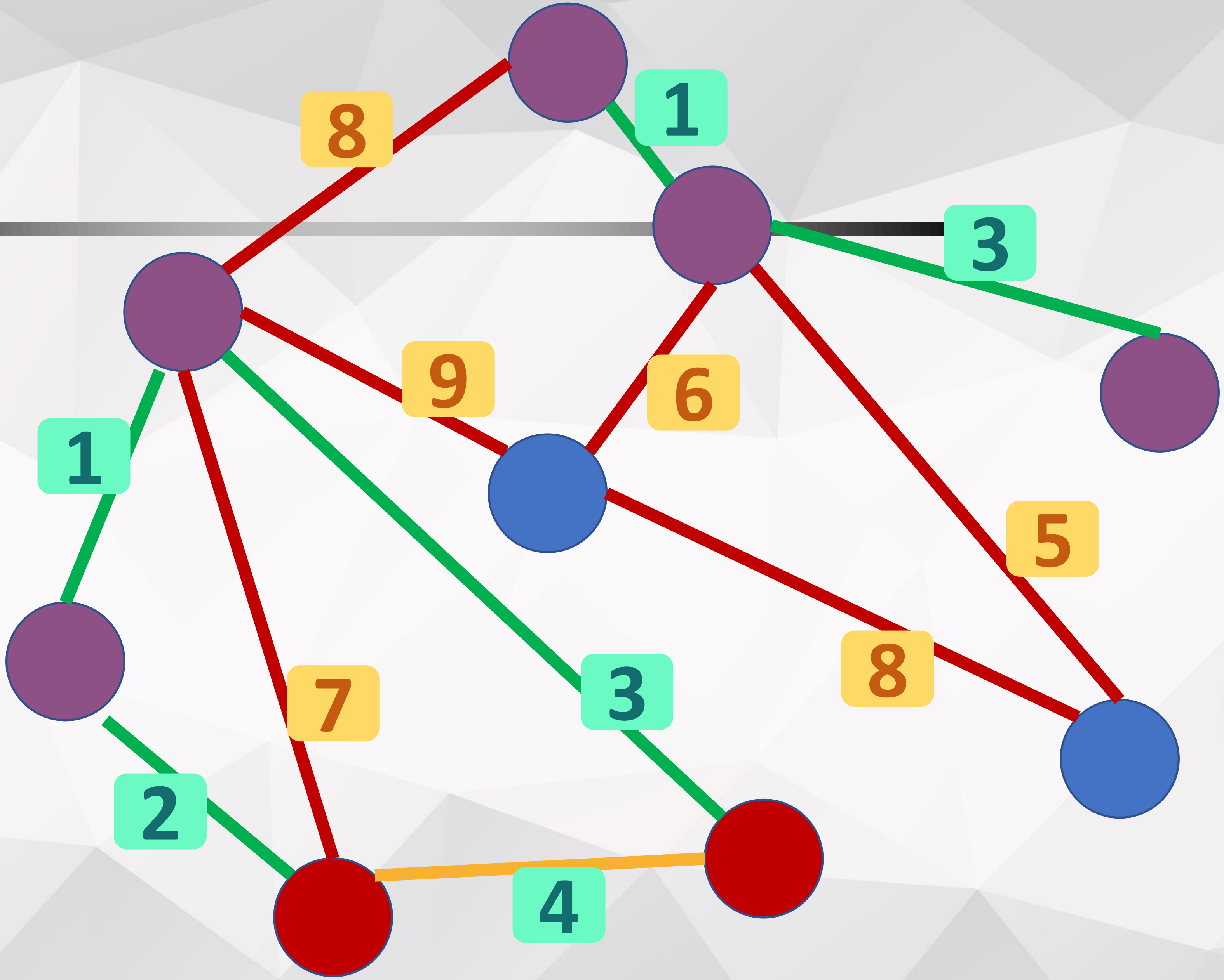

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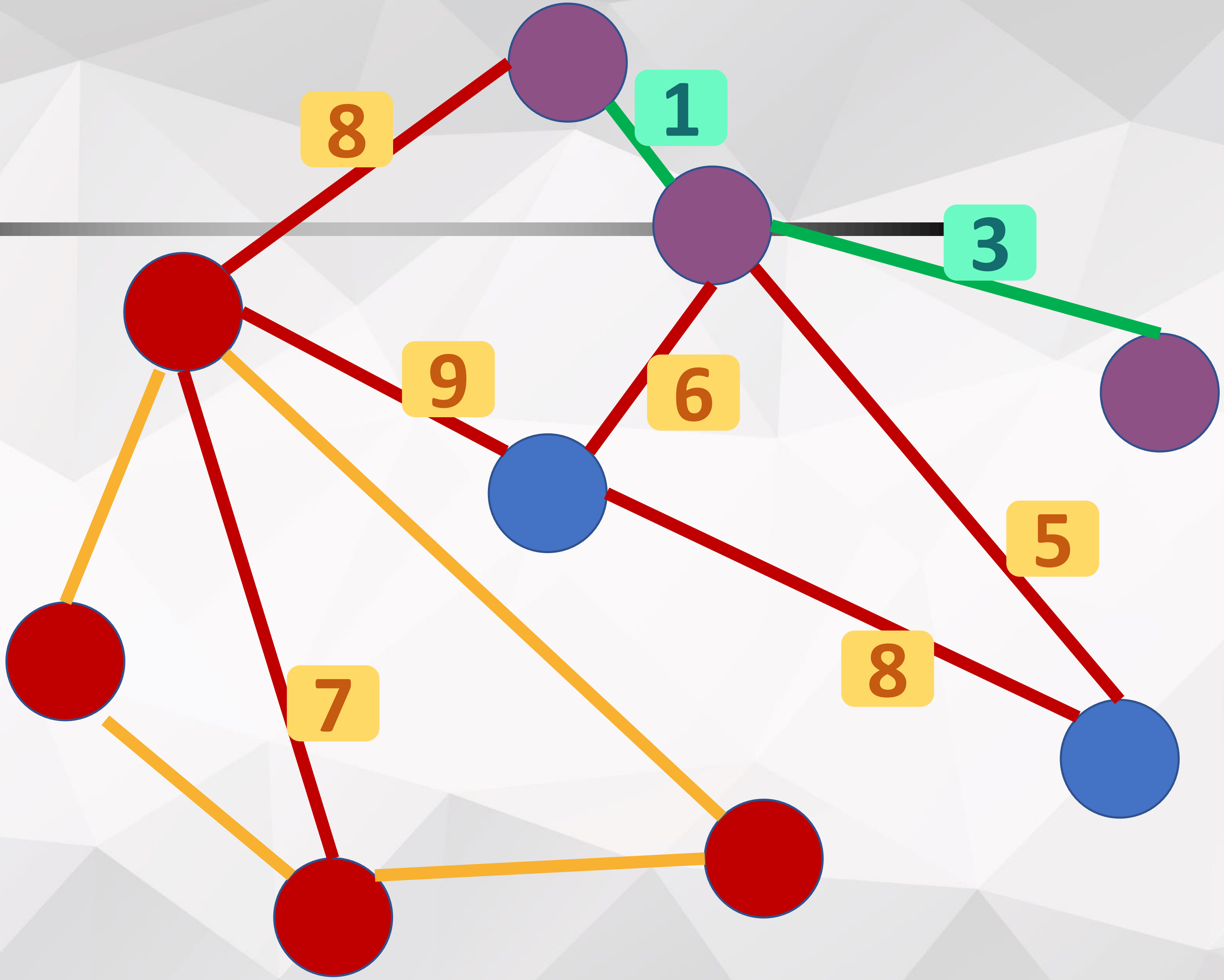

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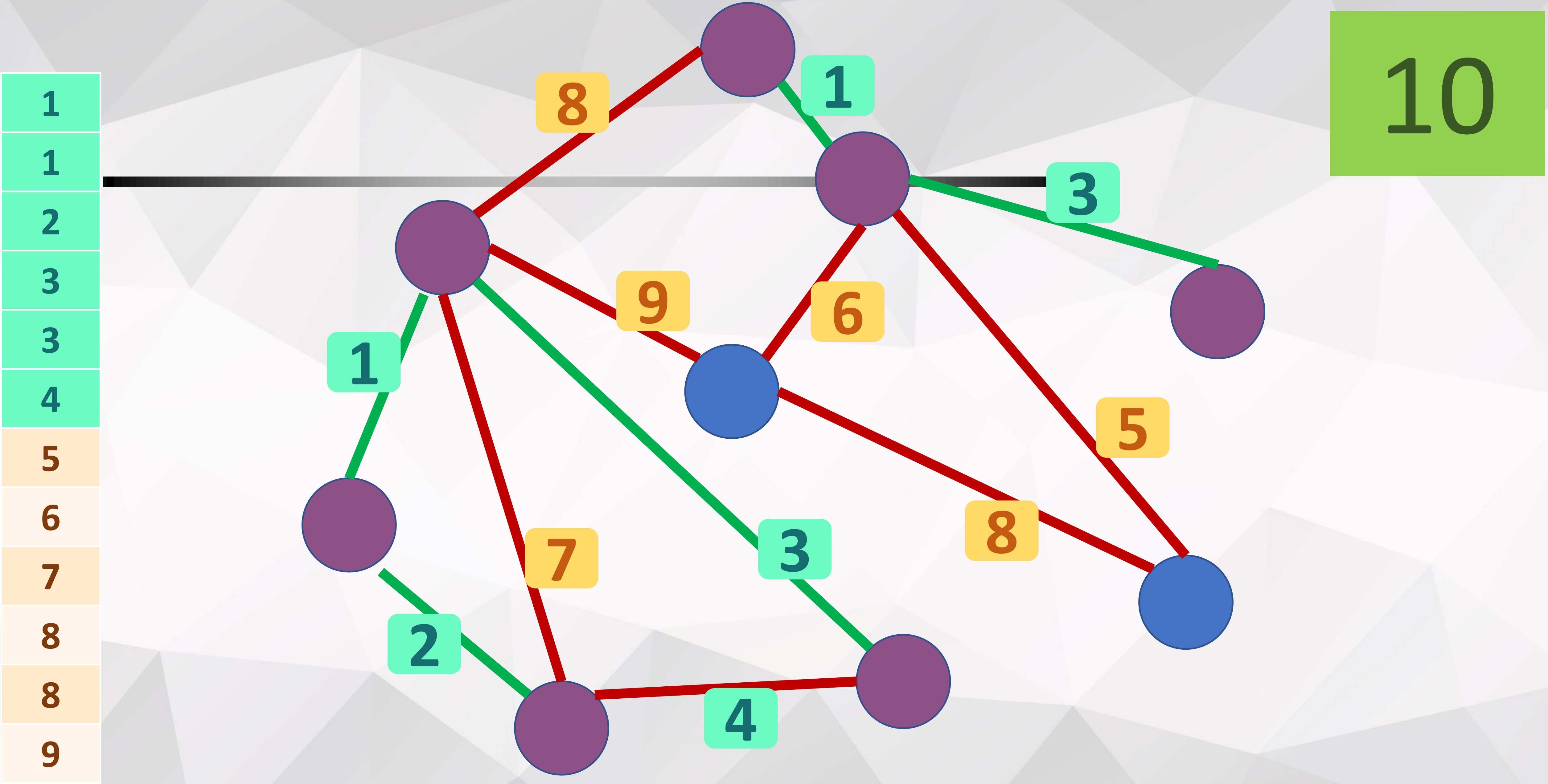

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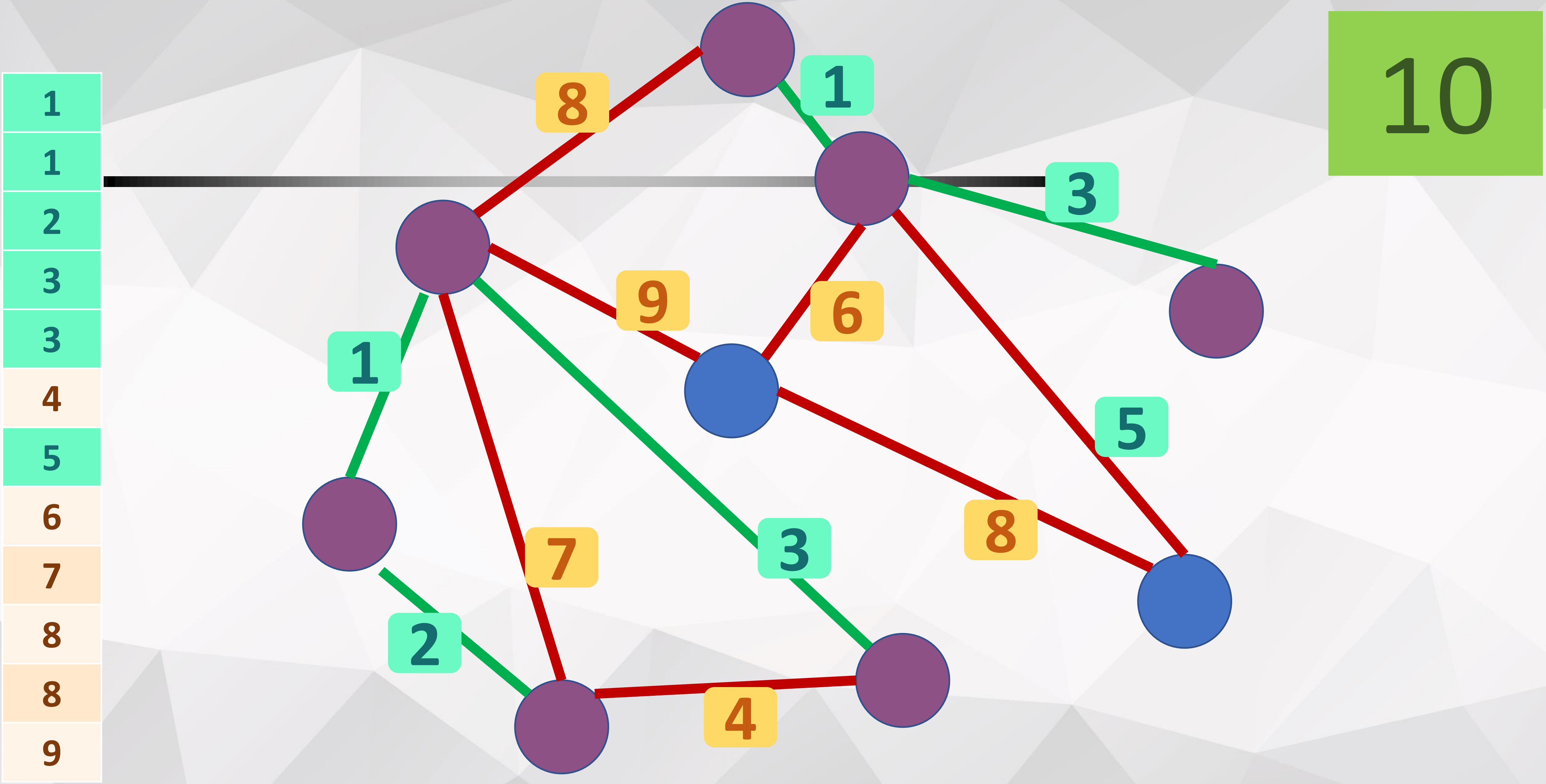

Competitive Programming

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檢查

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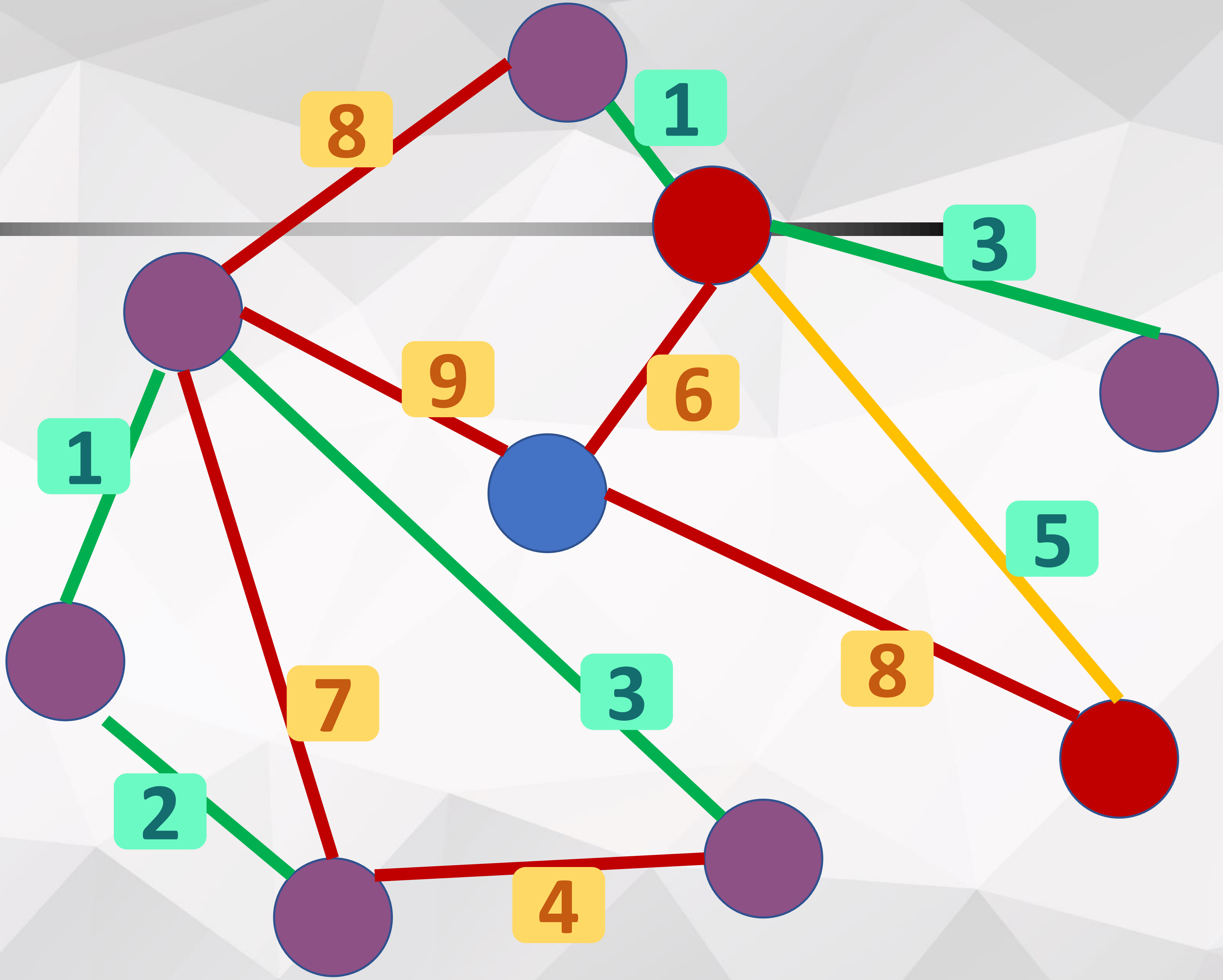

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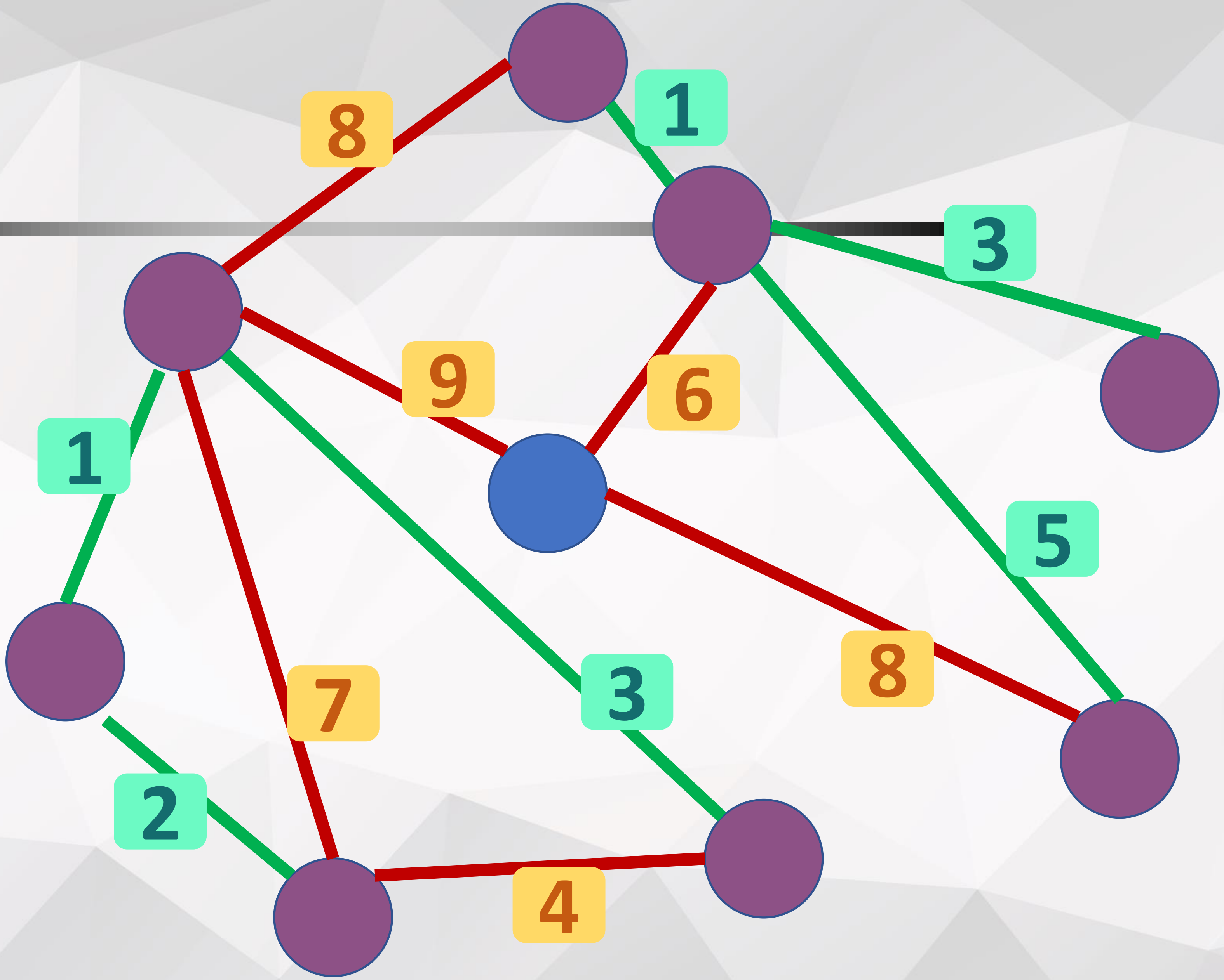

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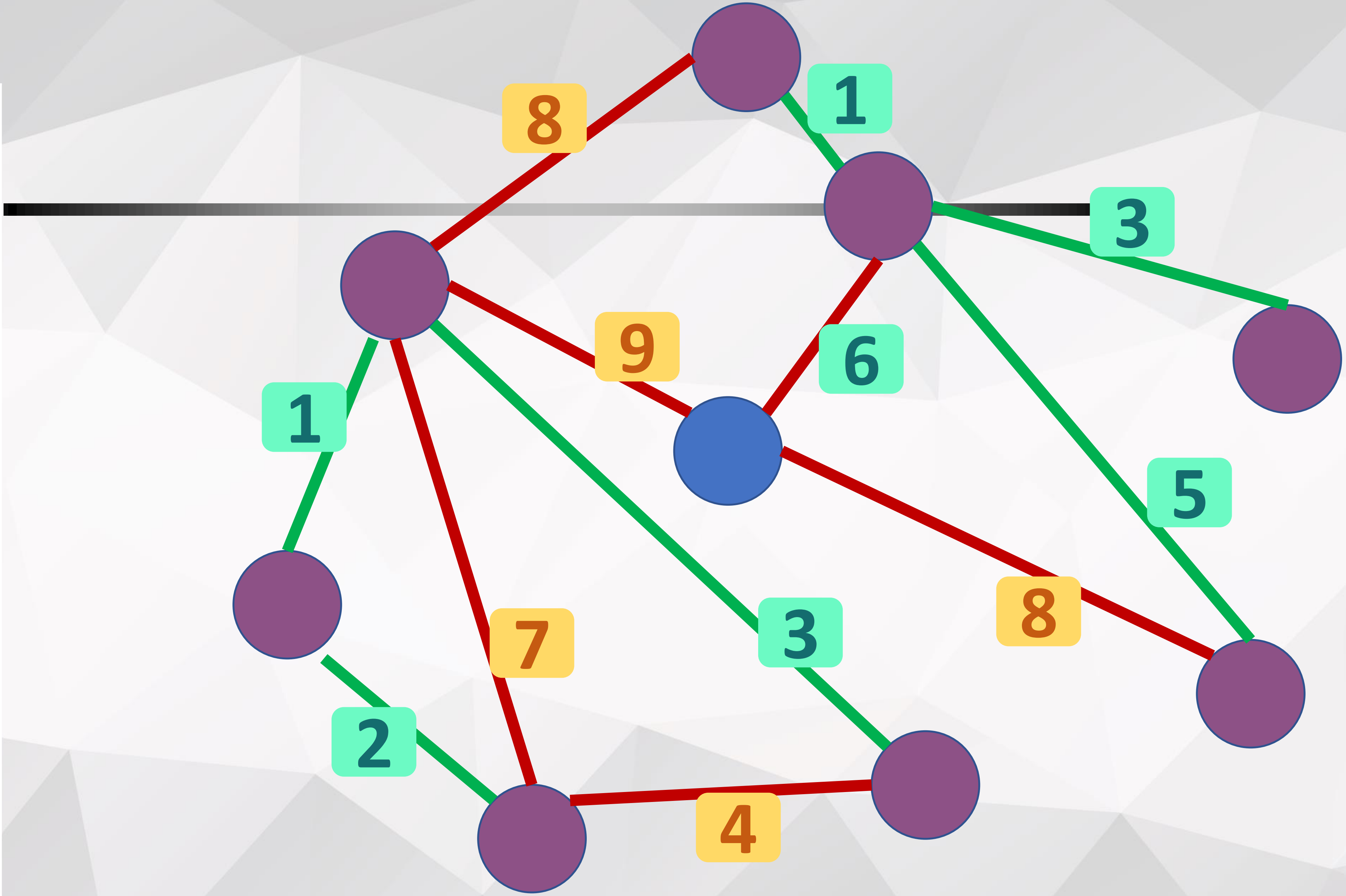

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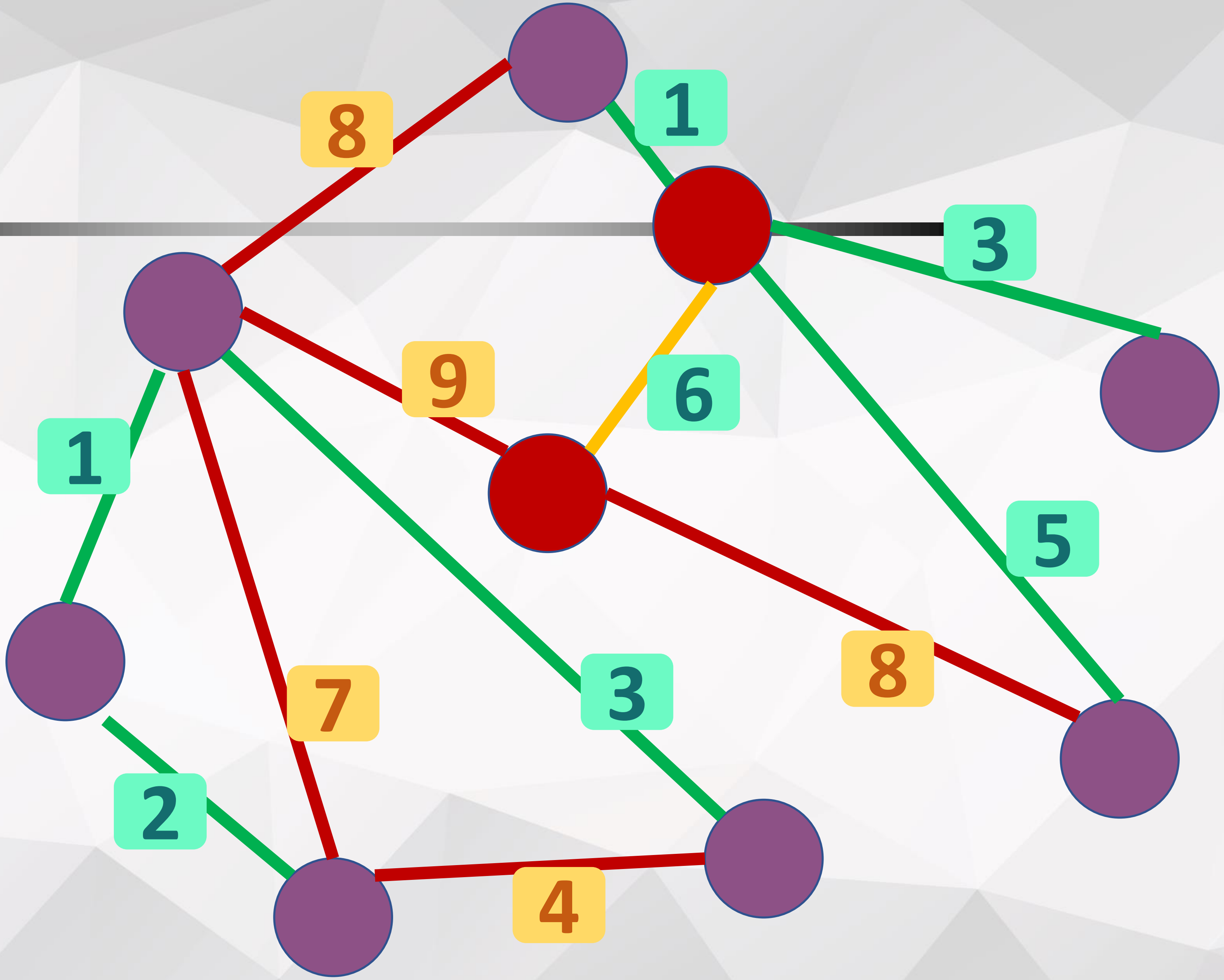

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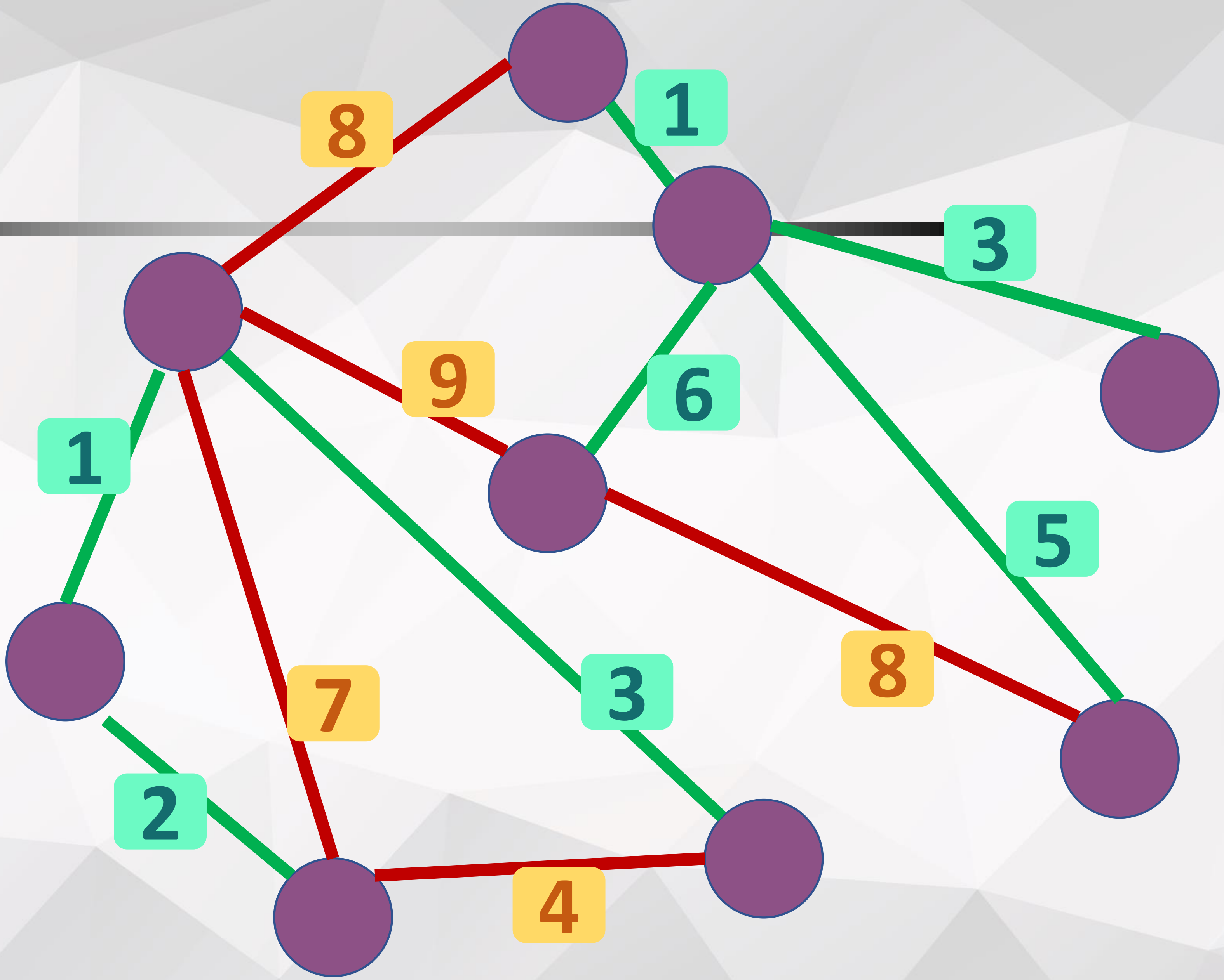

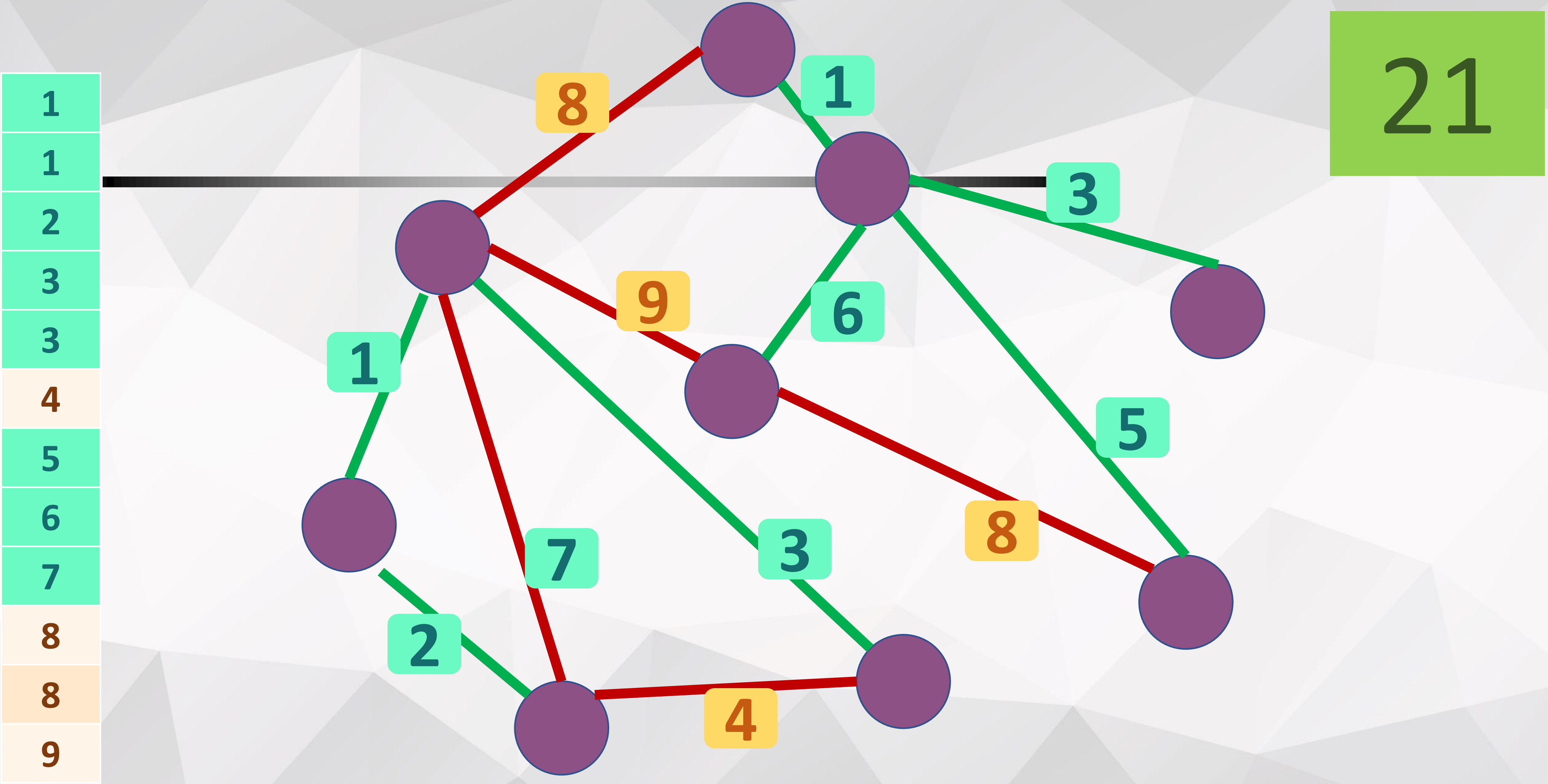
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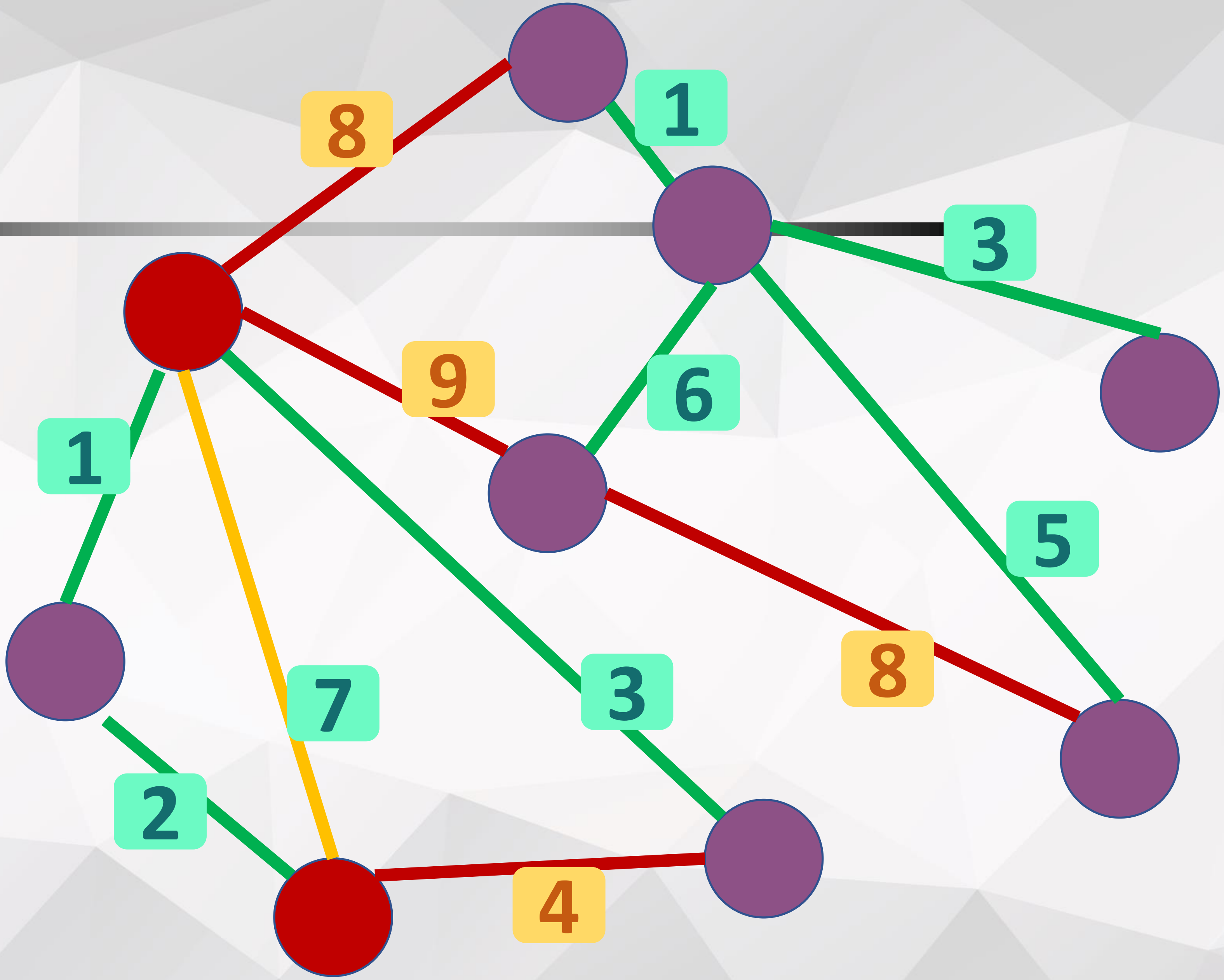

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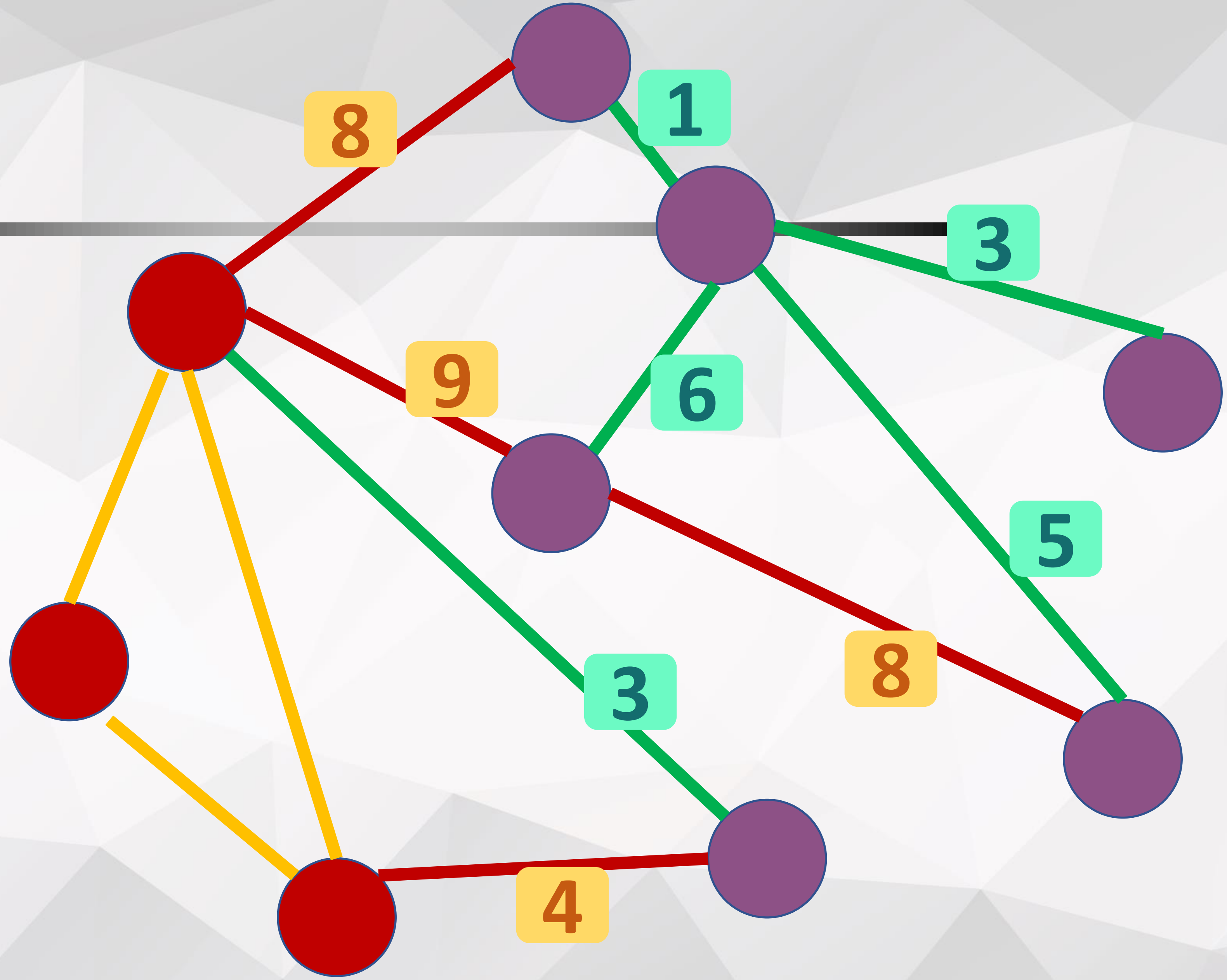

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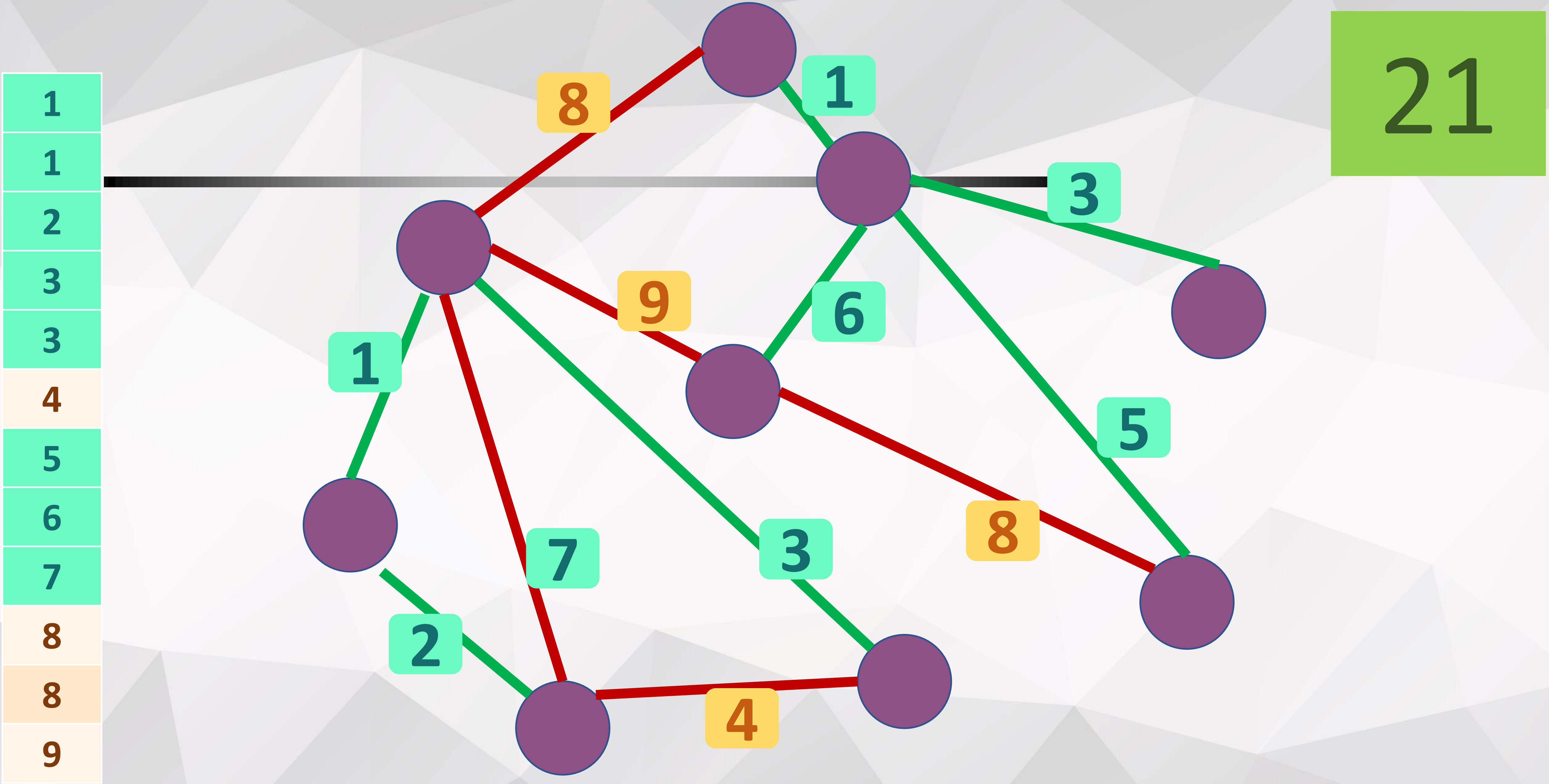
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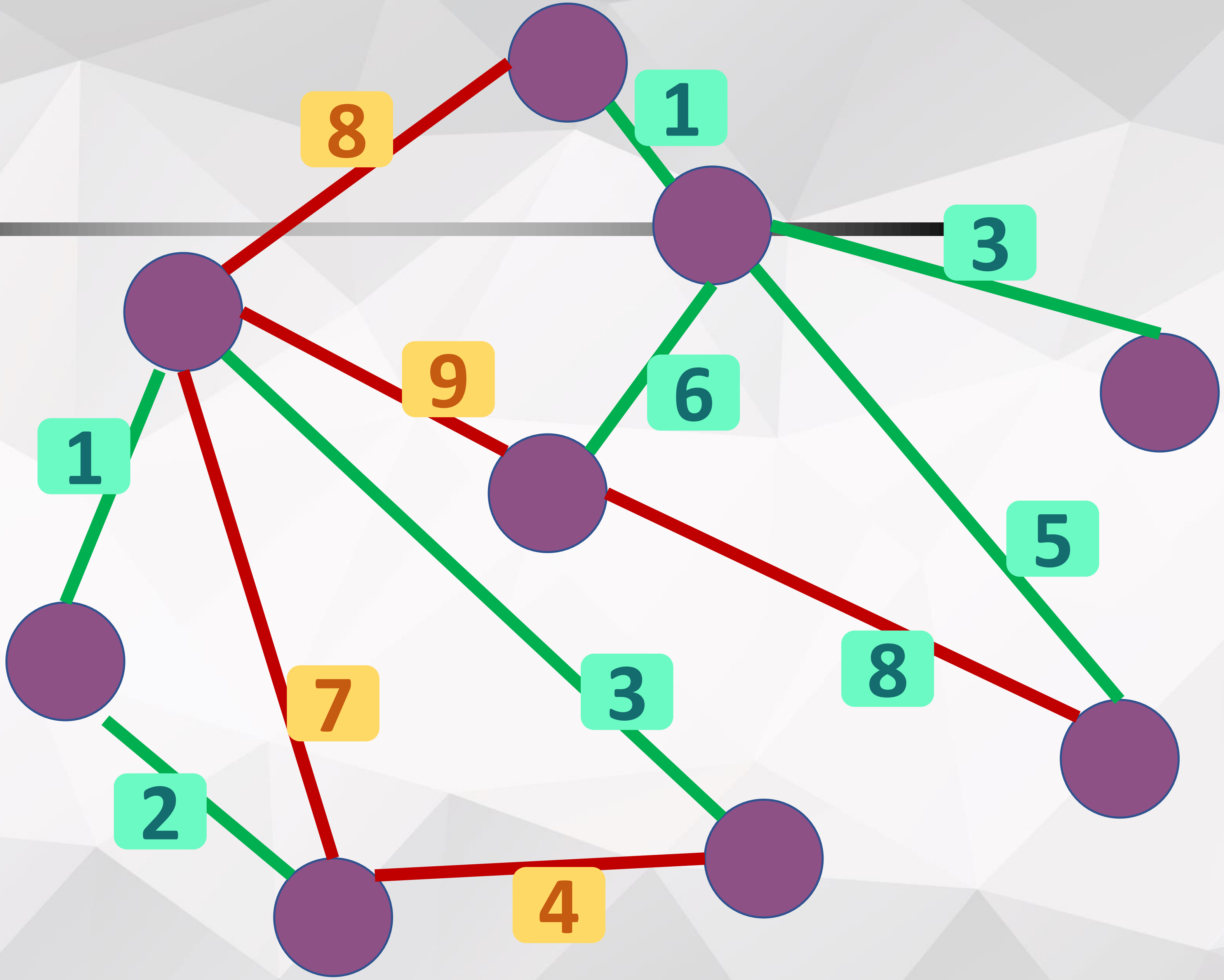

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Competitive Programming

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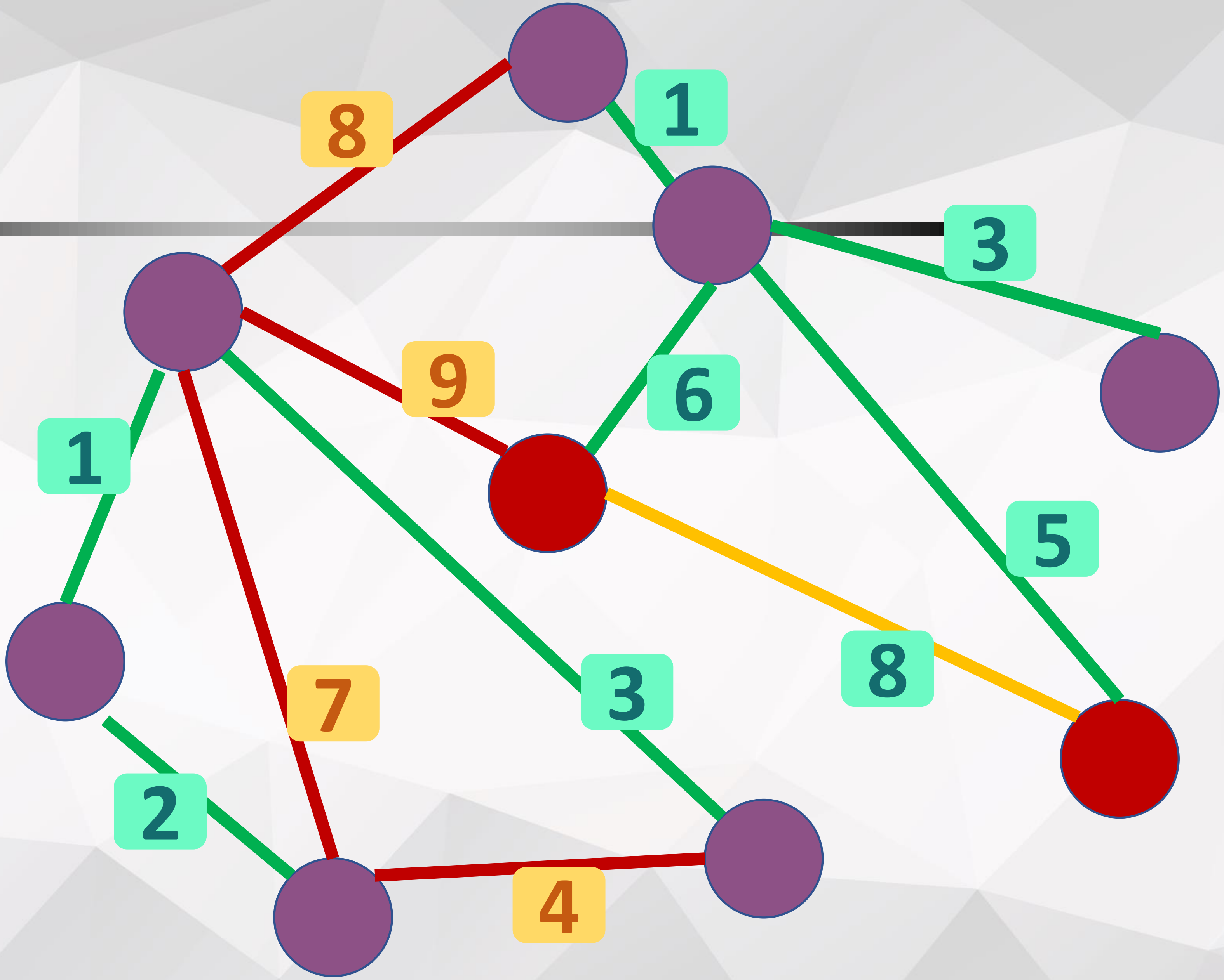

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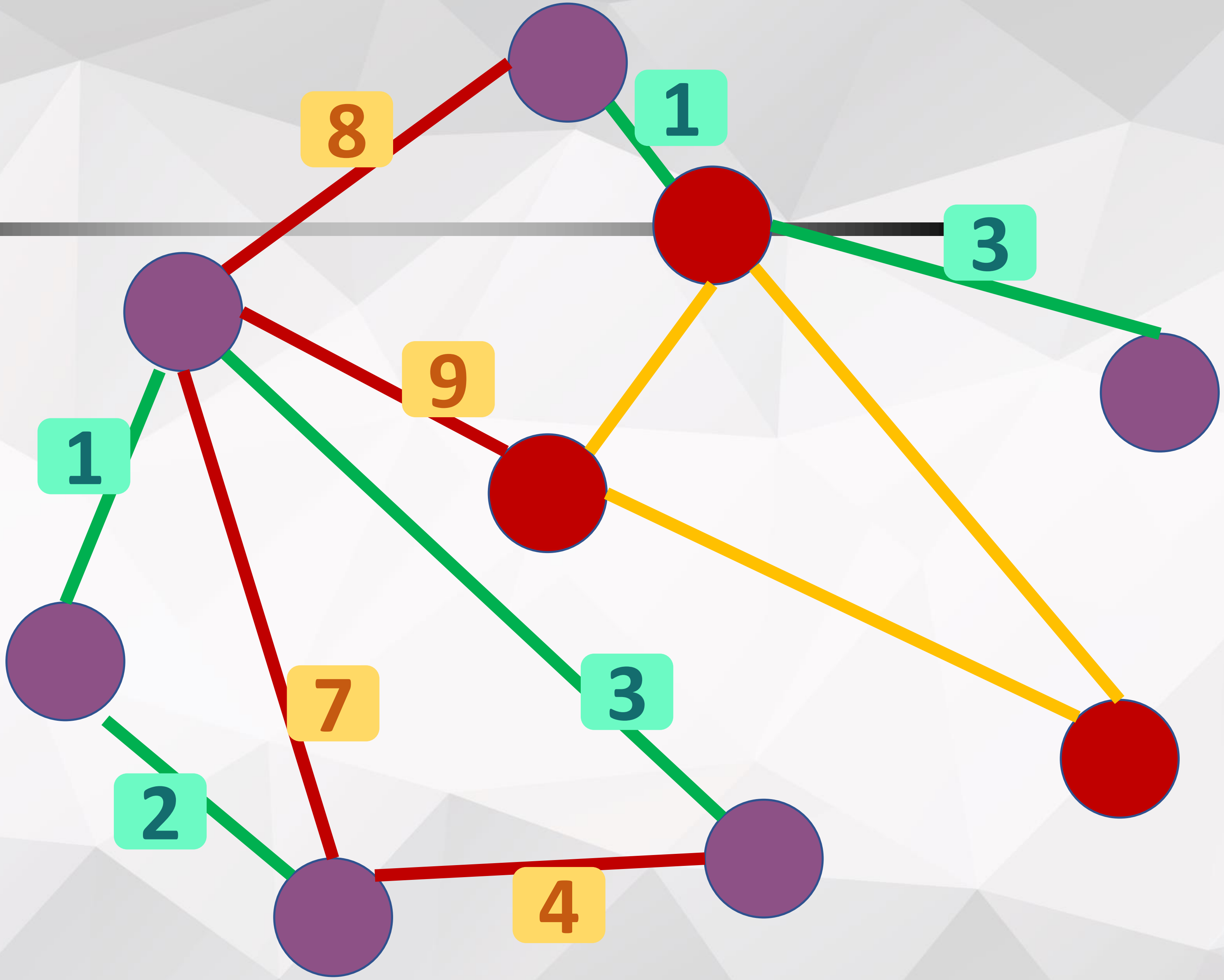

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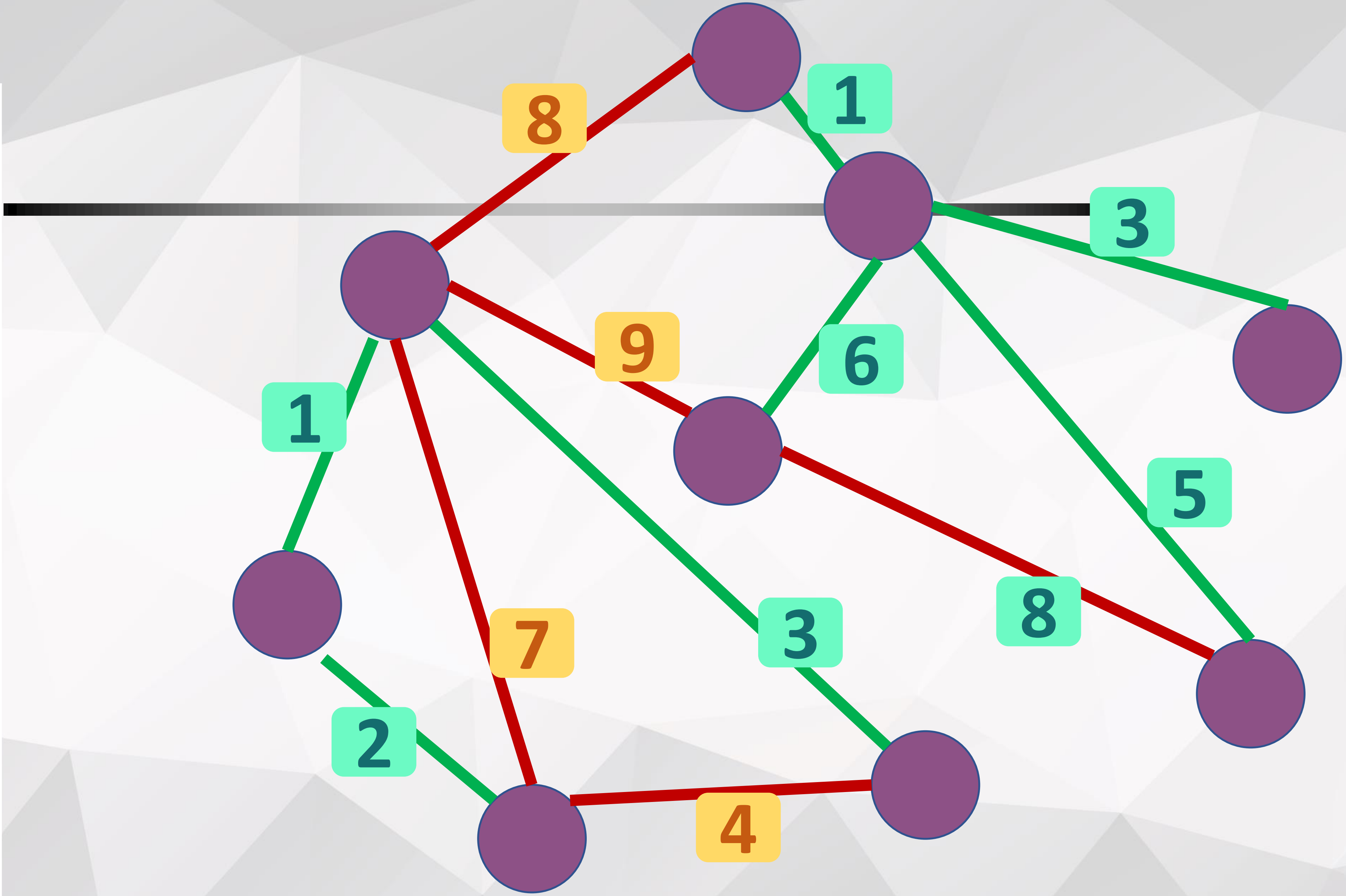

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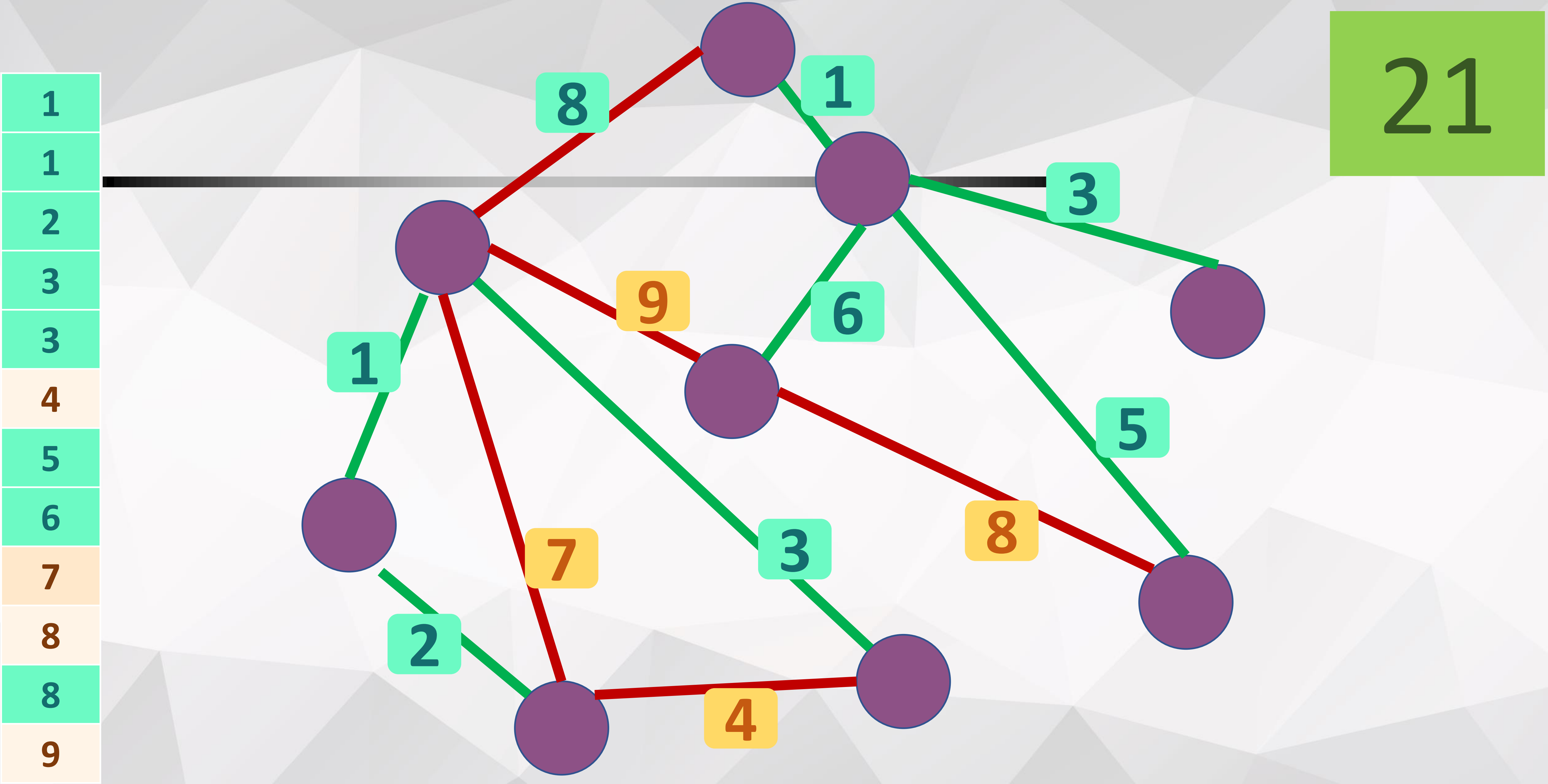
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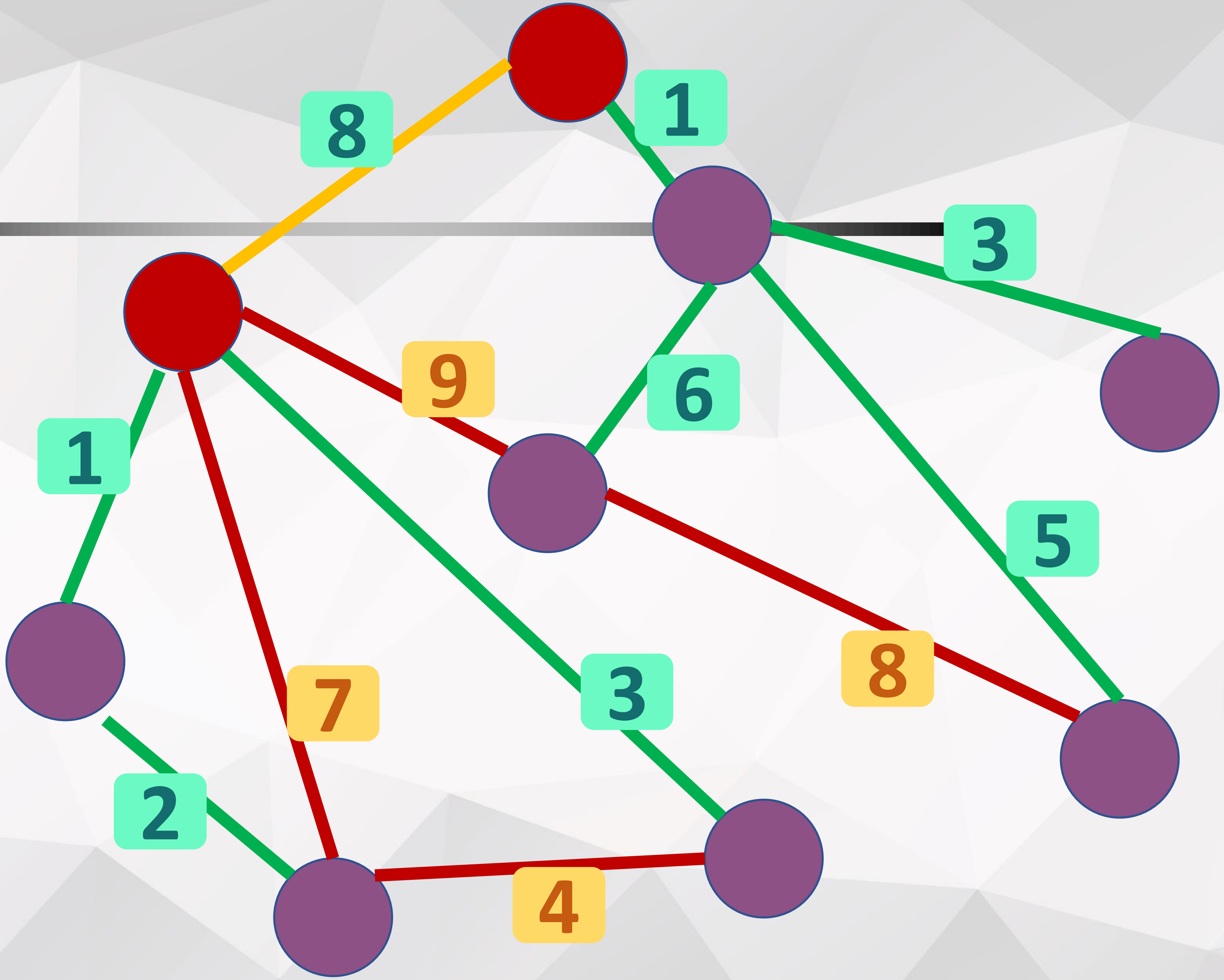

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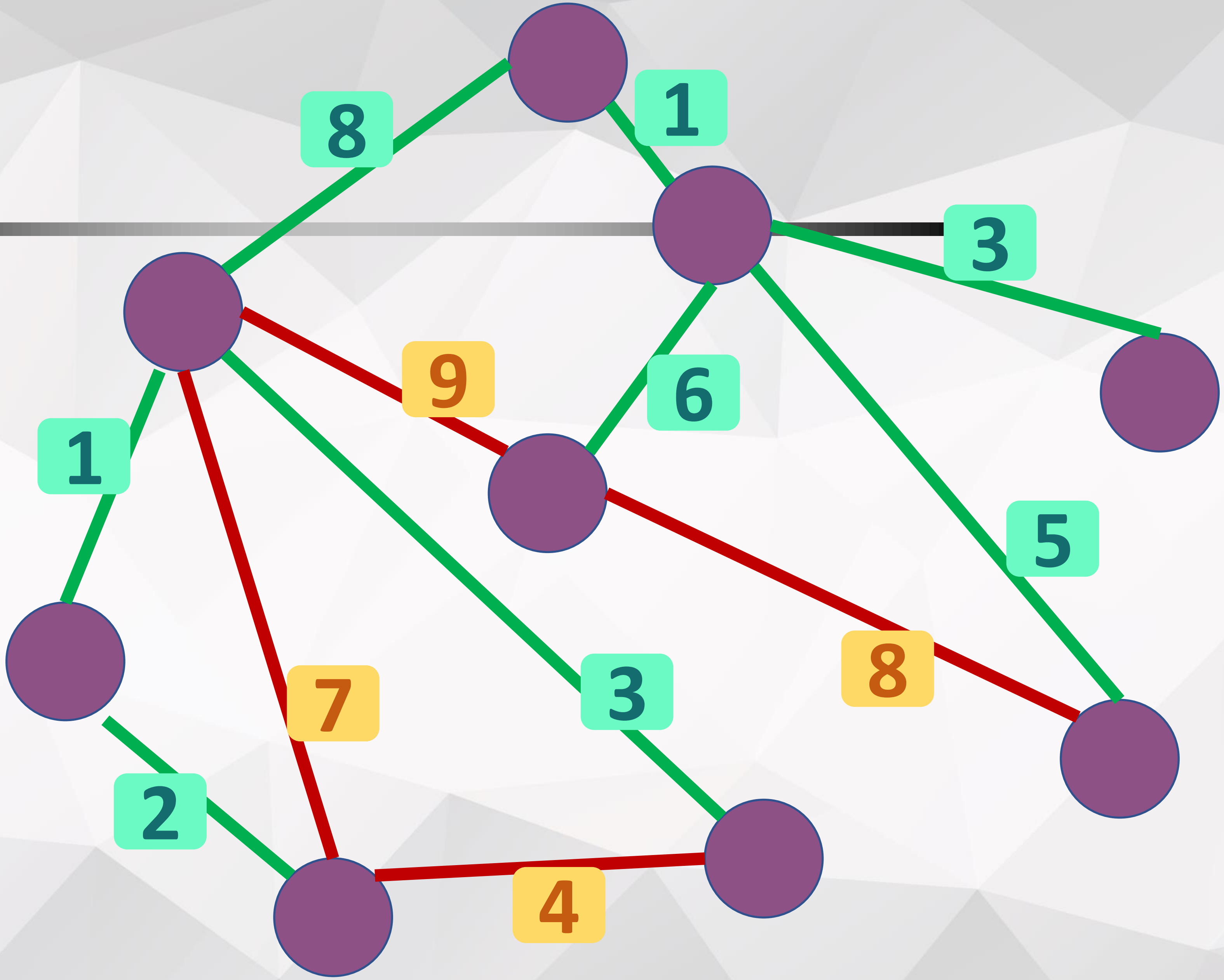

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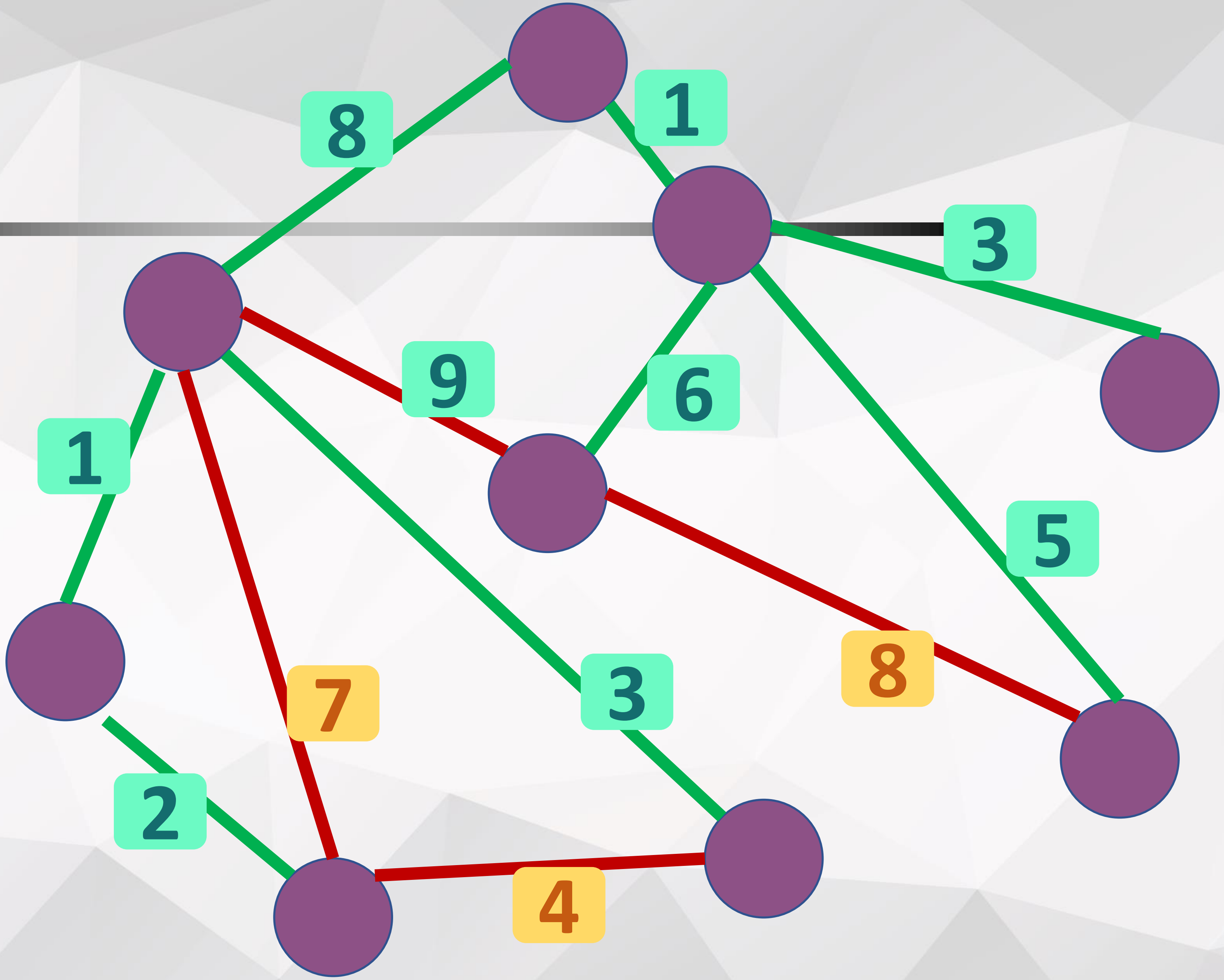

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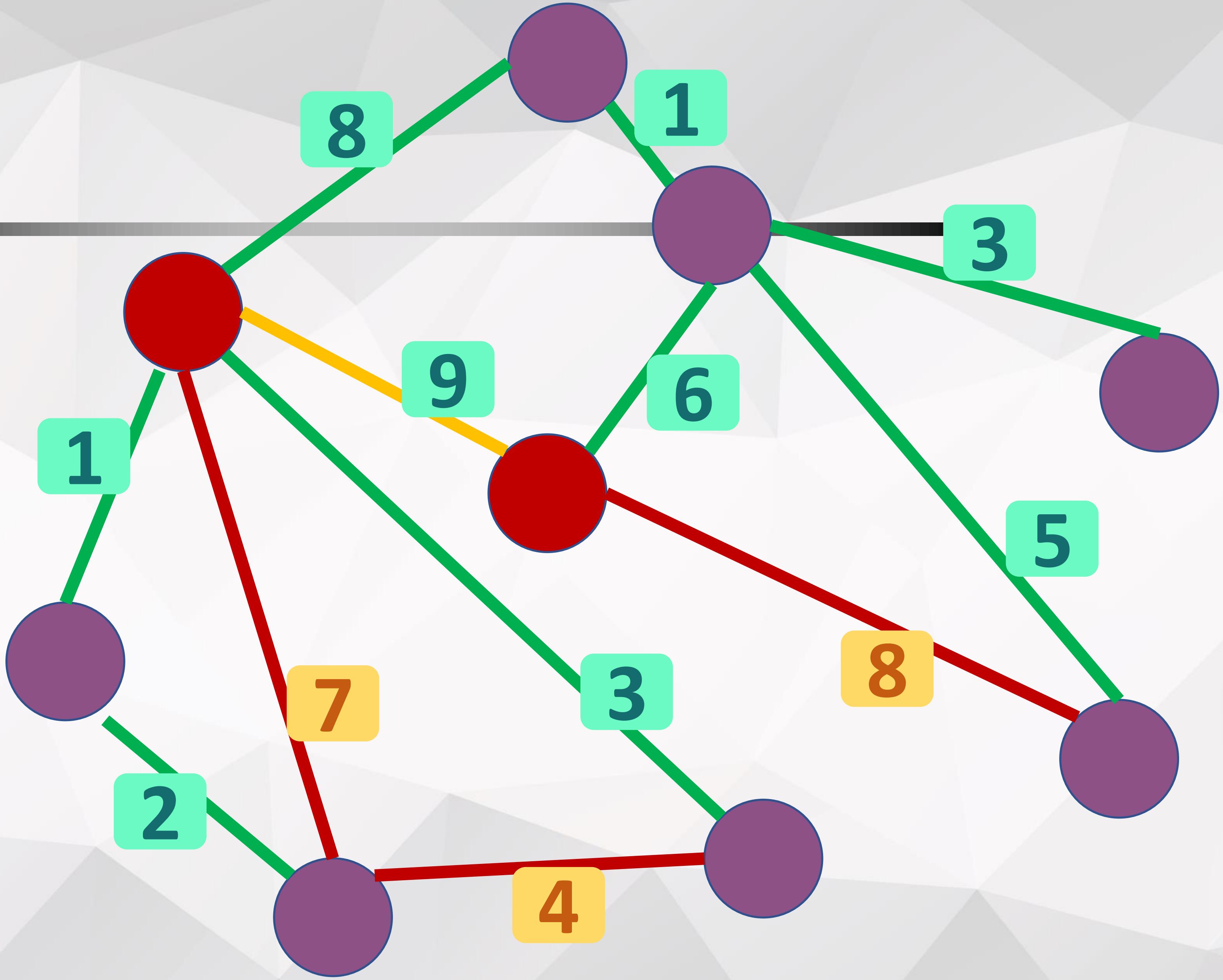

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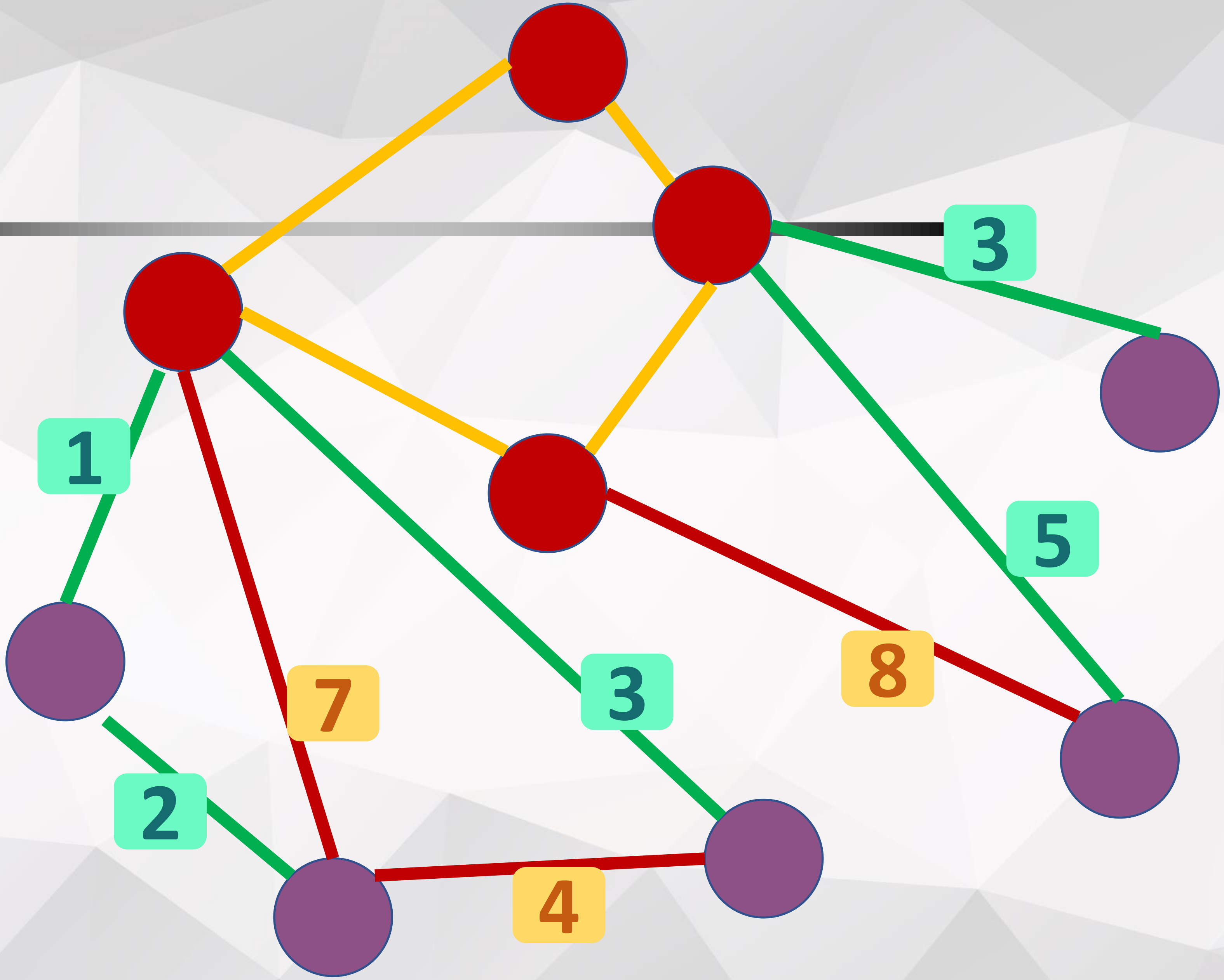

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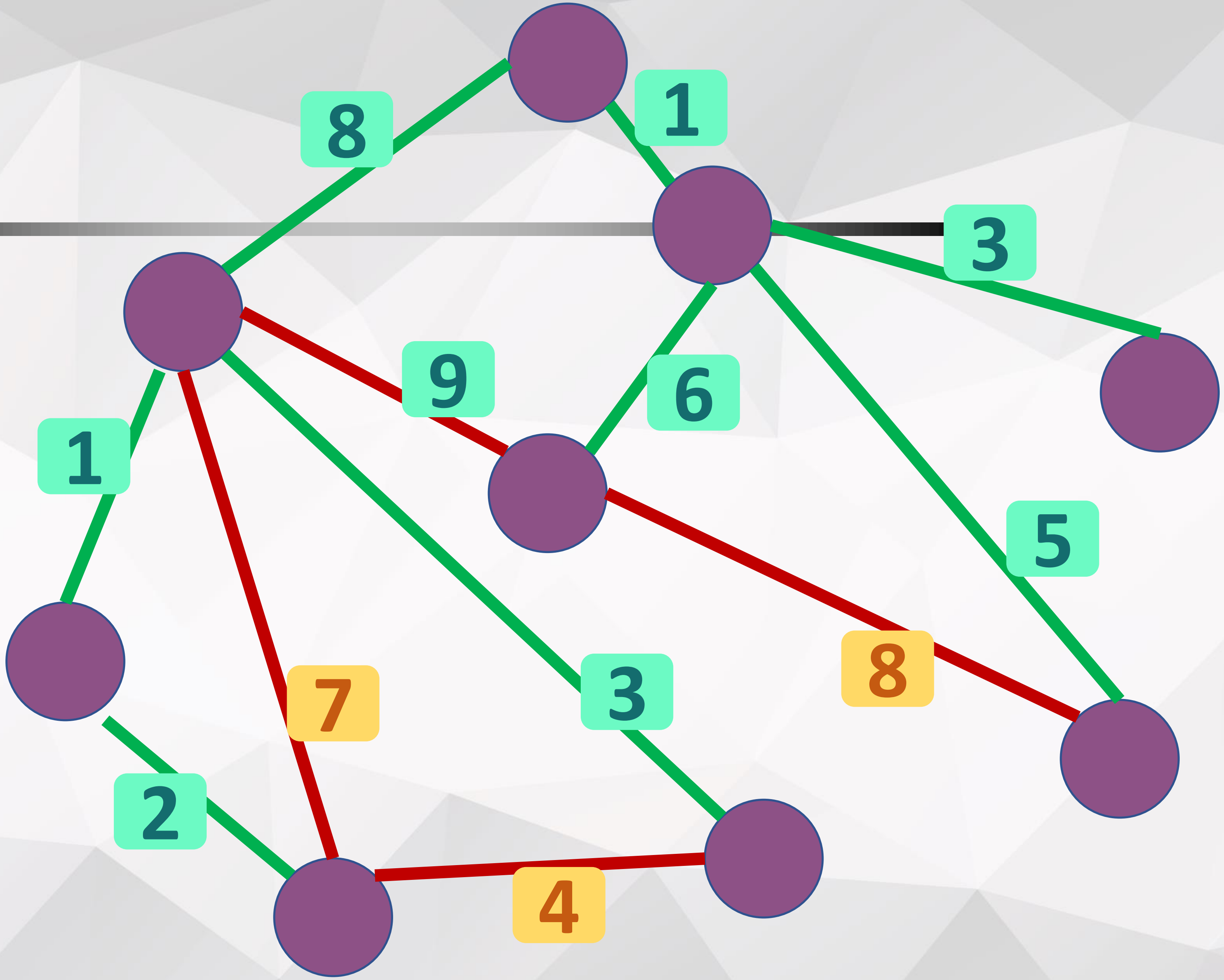

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Competitive Programming

遍歷完成

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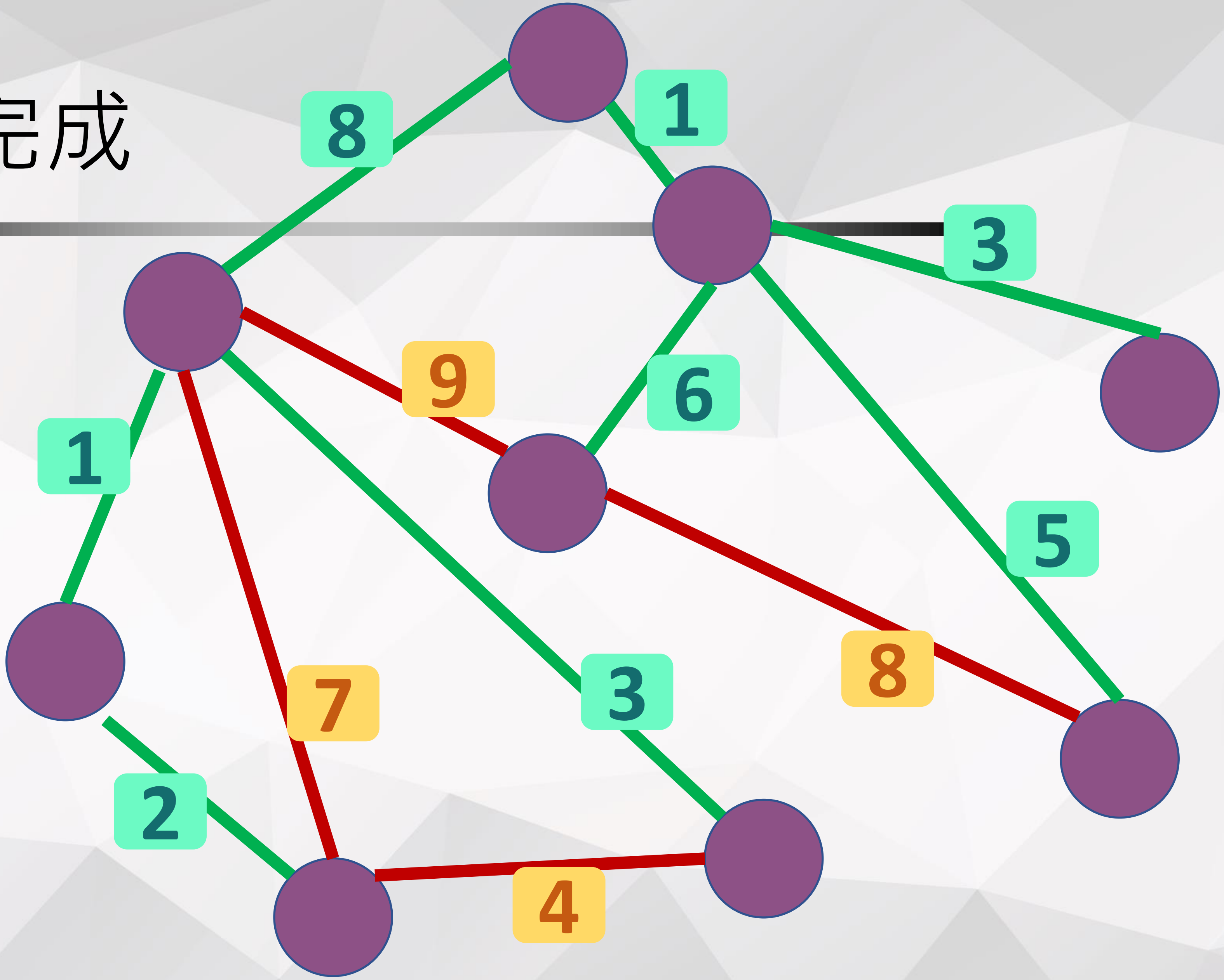

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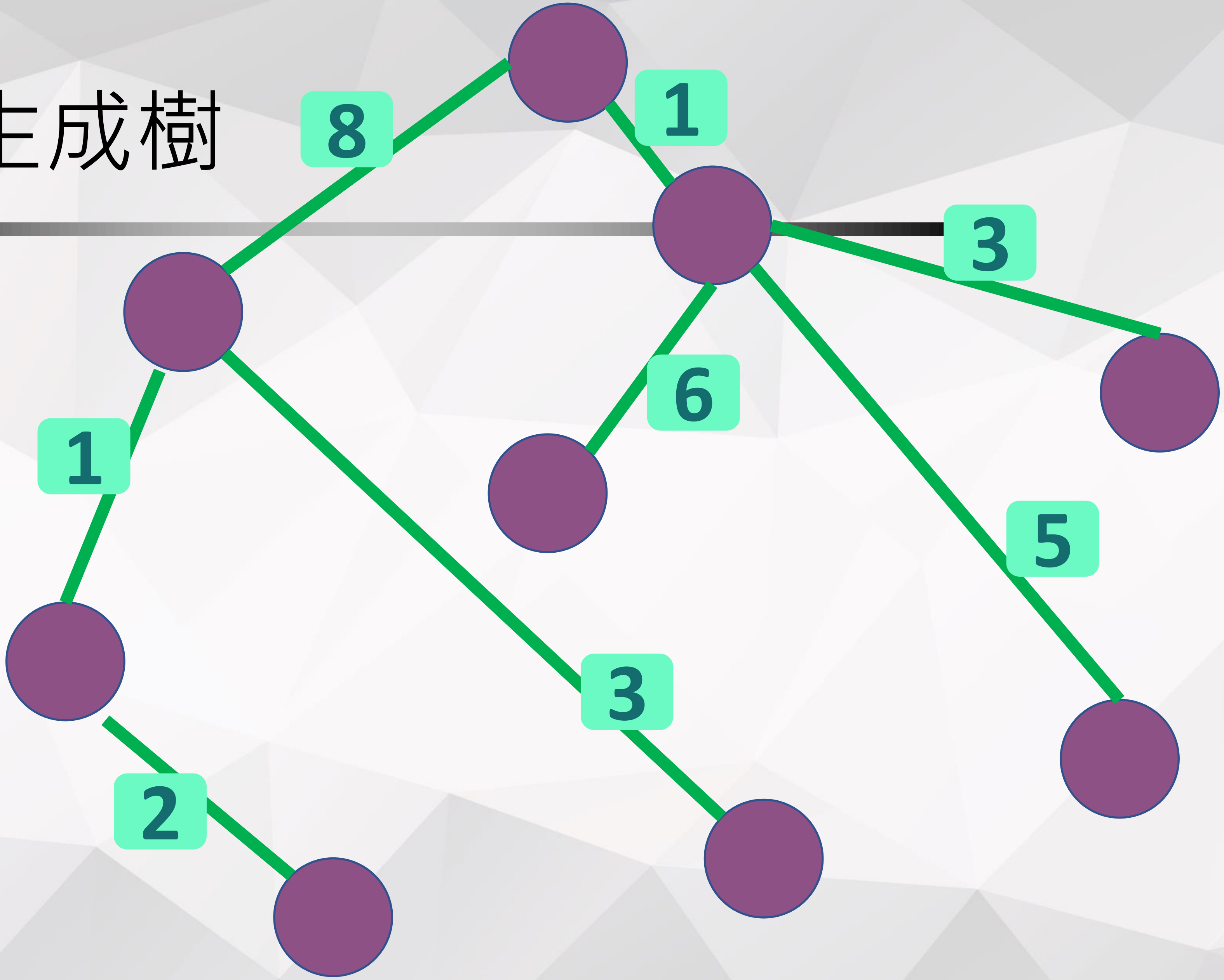
最小生成樹



Kruskal 實作

若使用 DFS 判斷兩點是否屬於同個連通塊

則最終複雜度為 $O(|E|^2)$

- 枚舉每個邊 $O(|E|)$
- DFS 將連通塊上的邊都拜訪 $O(|E|)$

Kruskal 實作

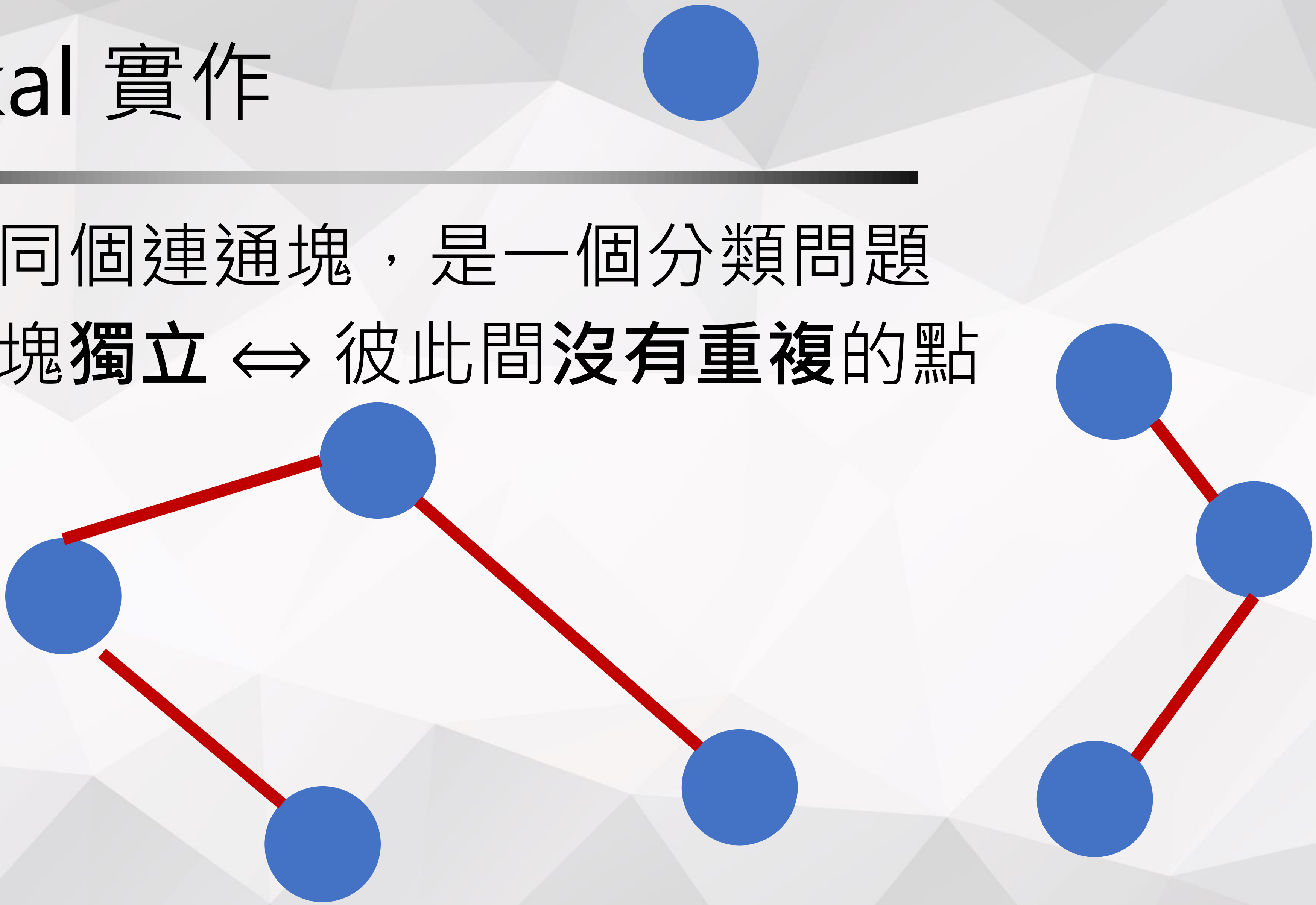
是否為同個連通塊，是一個分類問題

兩連通塊獨立 $\Leftrightarrow$ 彼此間沒有重複的點

Kruskal 實作

是否為同個連通塊，是一個分類問題

兩連通塊**獨立** ⇔ 彼此間**沒有重複**的點



Kruskal 實作

是否為同個連通塊，是一個分類問題
兩連通塊獨立 ⟺ 彼此間沒有重複的點

也就是說，連通塊們是個 Disjoint Sets
可以用 Union-Find Forest 改善複雜度
第五週教材中有 Union-Find Forest 的介紹

Kruskal 實作

```
bool cmp(const edge &A, const edge &B)
   { return A.w < B.w; }
```

Kruskal 實作

```
vector<edge> E; // 邊集合
  ⋮
sort(E.begin(), E.end(), cmp);

for (edge e: E) {
  int a = Find(e.u), b = Find(e.v);
  if (a != b) {
    Union(e.u, e.v);
    cost += E.w;
    MST.emplace_back(u, v, w);
  }
}
```

Kruskal 實作

用 Union-Find Forest 改善複雜度

複雜度為 $O(|E|\log_2|E| + |E|\cdot \alpha)$

- α 為 Union-Find Forest 的時間成本

Questions?

練習

- UVa OJ 10369 Arctic Network
- AIZU 1280 Slim Span

最小生成樹

- Kruskal 演算法
- Prim 演算法

Prim 演算法

很類似的

兩個重要前提

- 樹是**無環**的**連通**圖

- 若圖只有點無任何邊，那每點都是彼此獨立**連通塊**

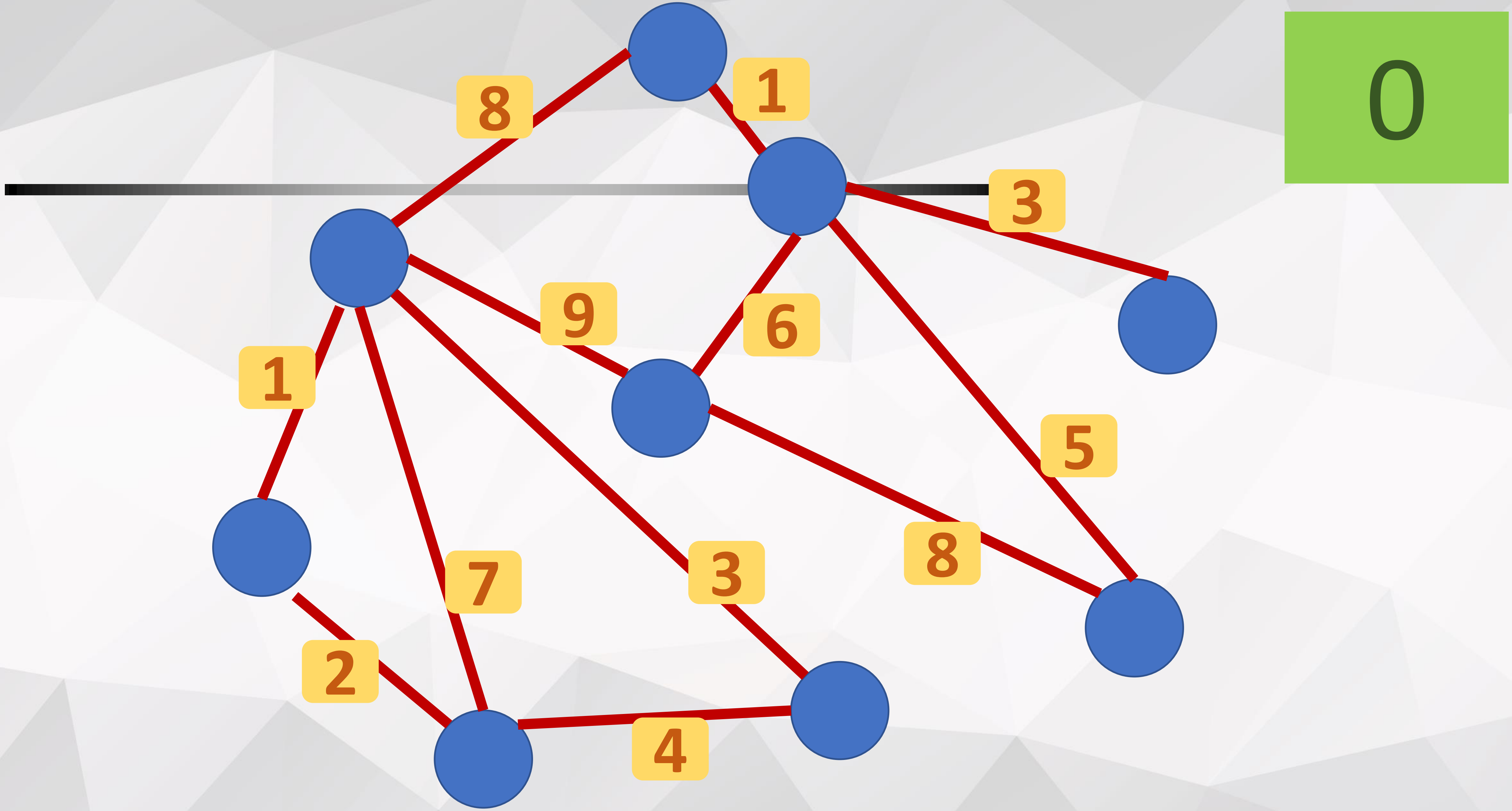
Prim 演算法

Prim 維護一個**未完成的生成樹**

Prim 演算法

Prim 維護一個未完成的生成樹

每次將樹**周遭**有**最小權重**的邊接到樹上，
使樹最終成長至最小生成樹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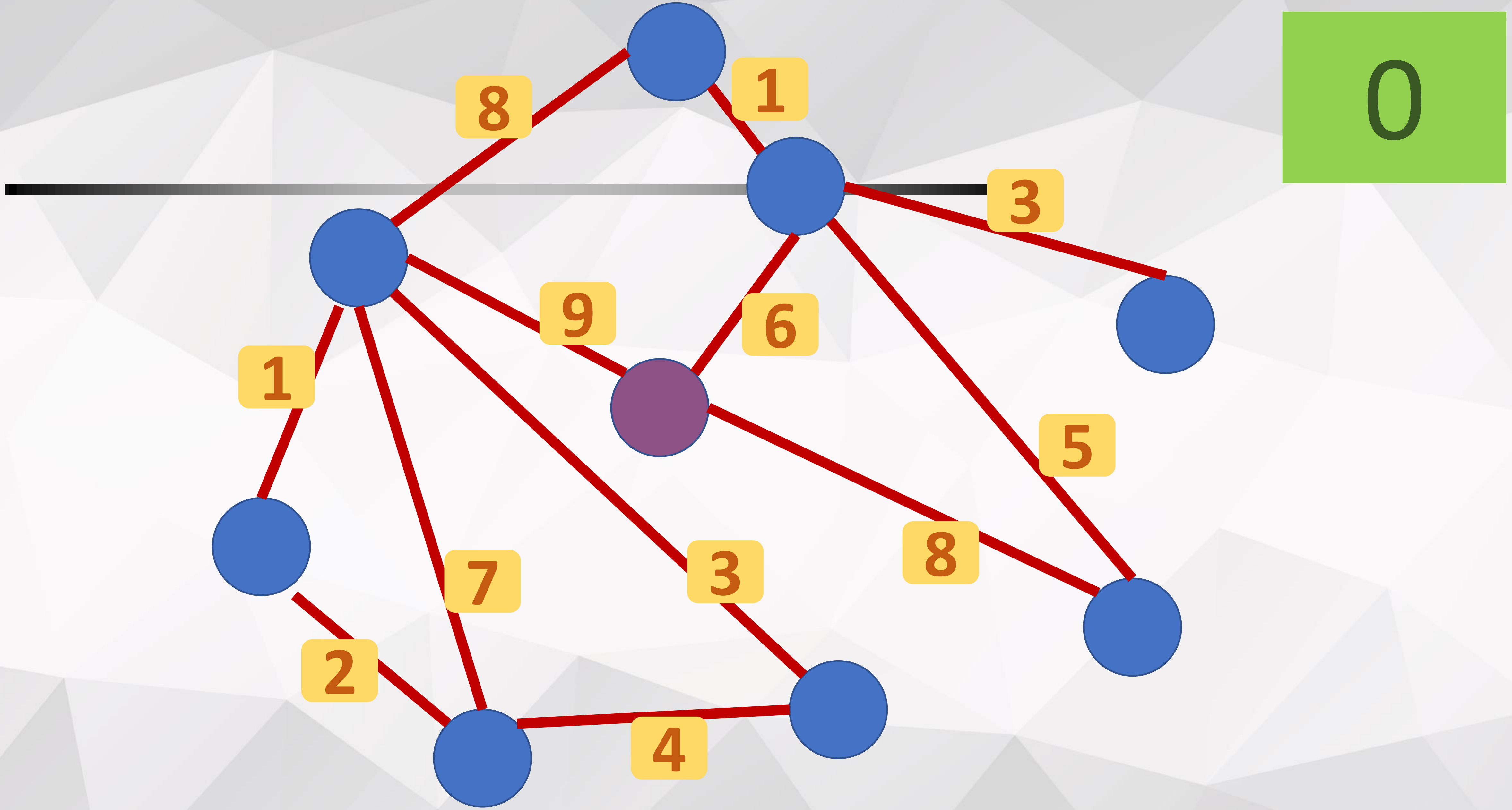
Prim 演算法

先隨便的挑任意點



Competitive Programming

Prim 演算法

先隨便的挑任意點

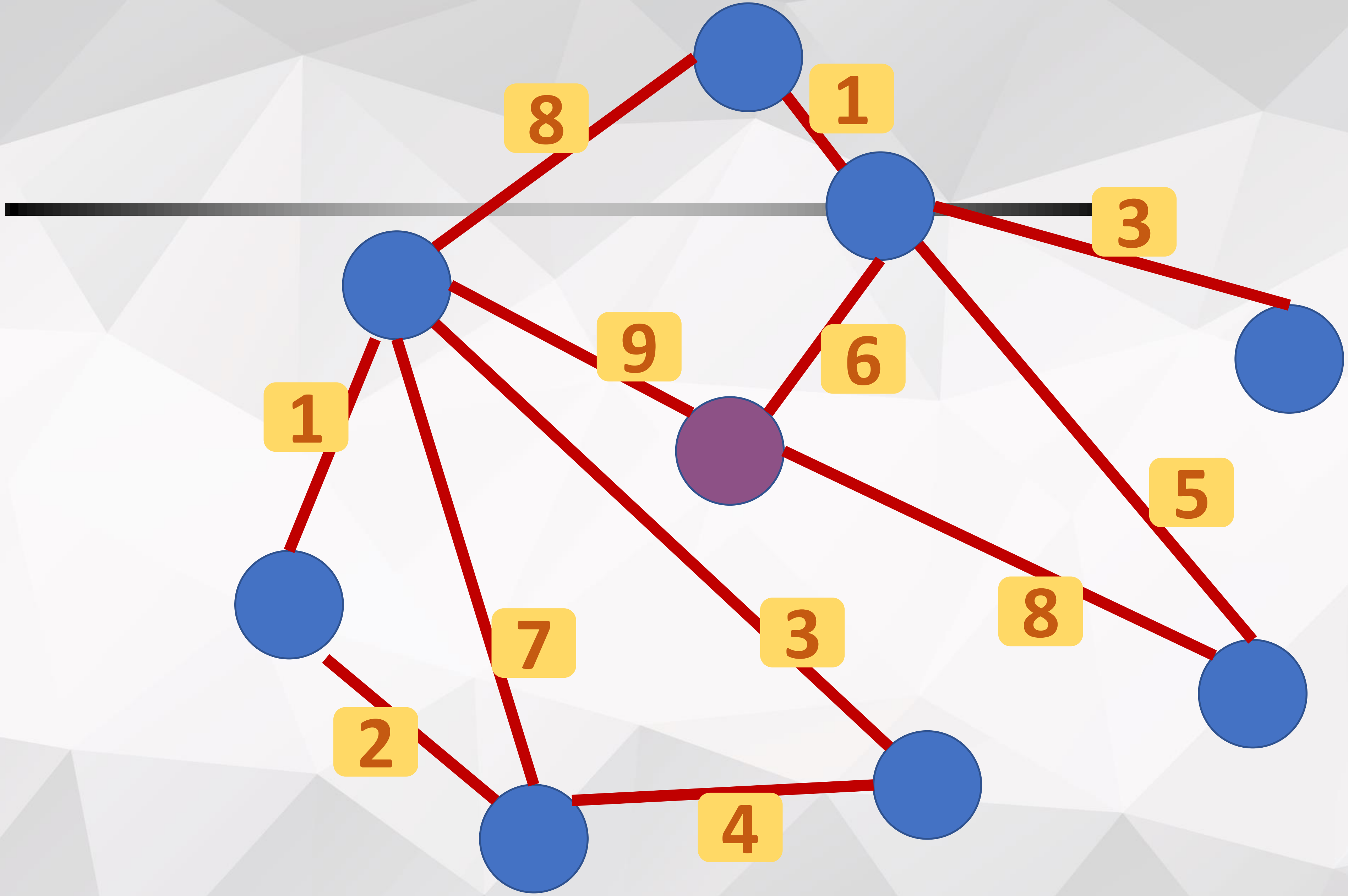
使它為初始的**未完成的生成樹**

Prim 演算法

先隨便的挑任意點

使它為初始的**未完成的生成樹**，稱它為 MST

明顯的，它是個無環連通圖



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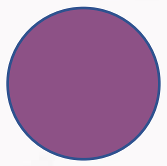
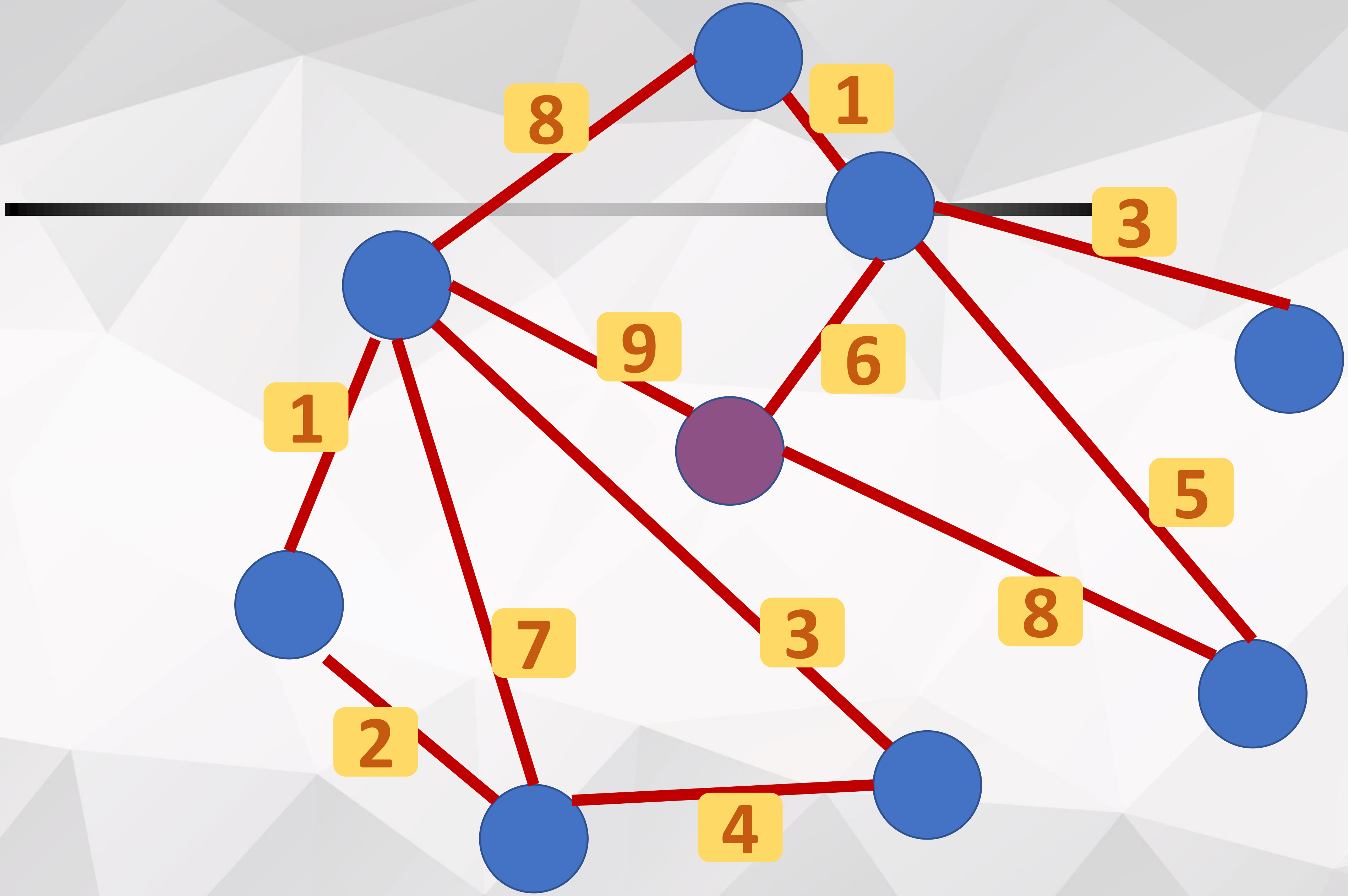


Prim 演算法

直覺的，每次將 MST 周遭權重最小的邊接上去

那麼最終產生的生成樹為最小生成樹



0

Competitive Programming

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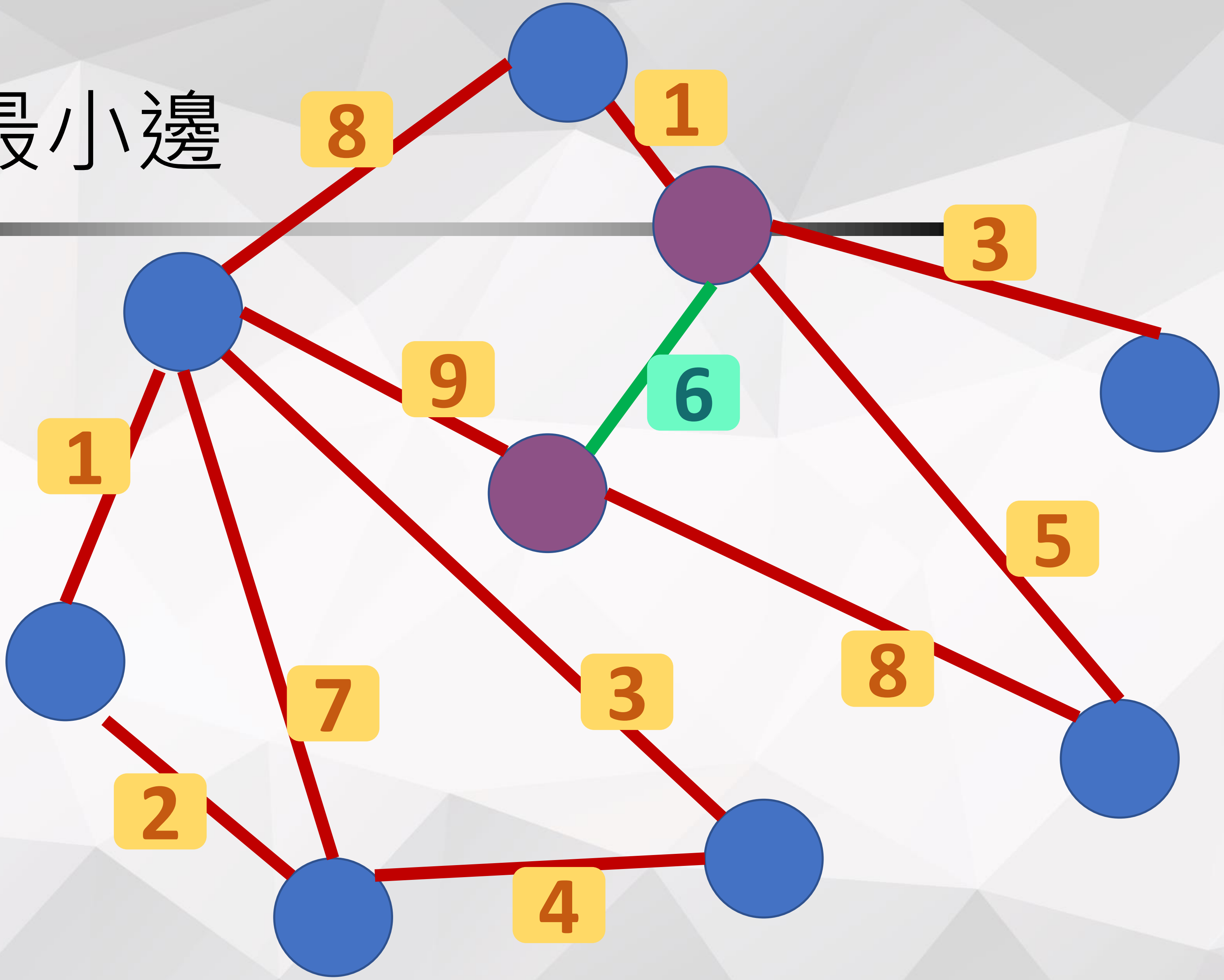
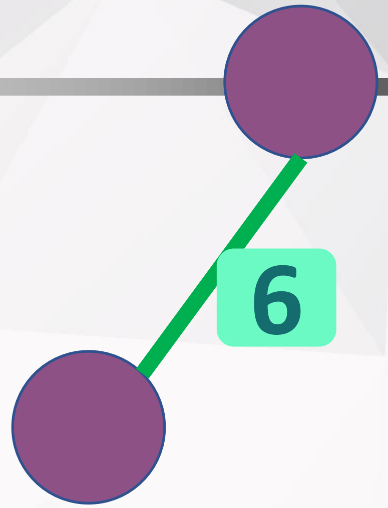
周圍最小邊

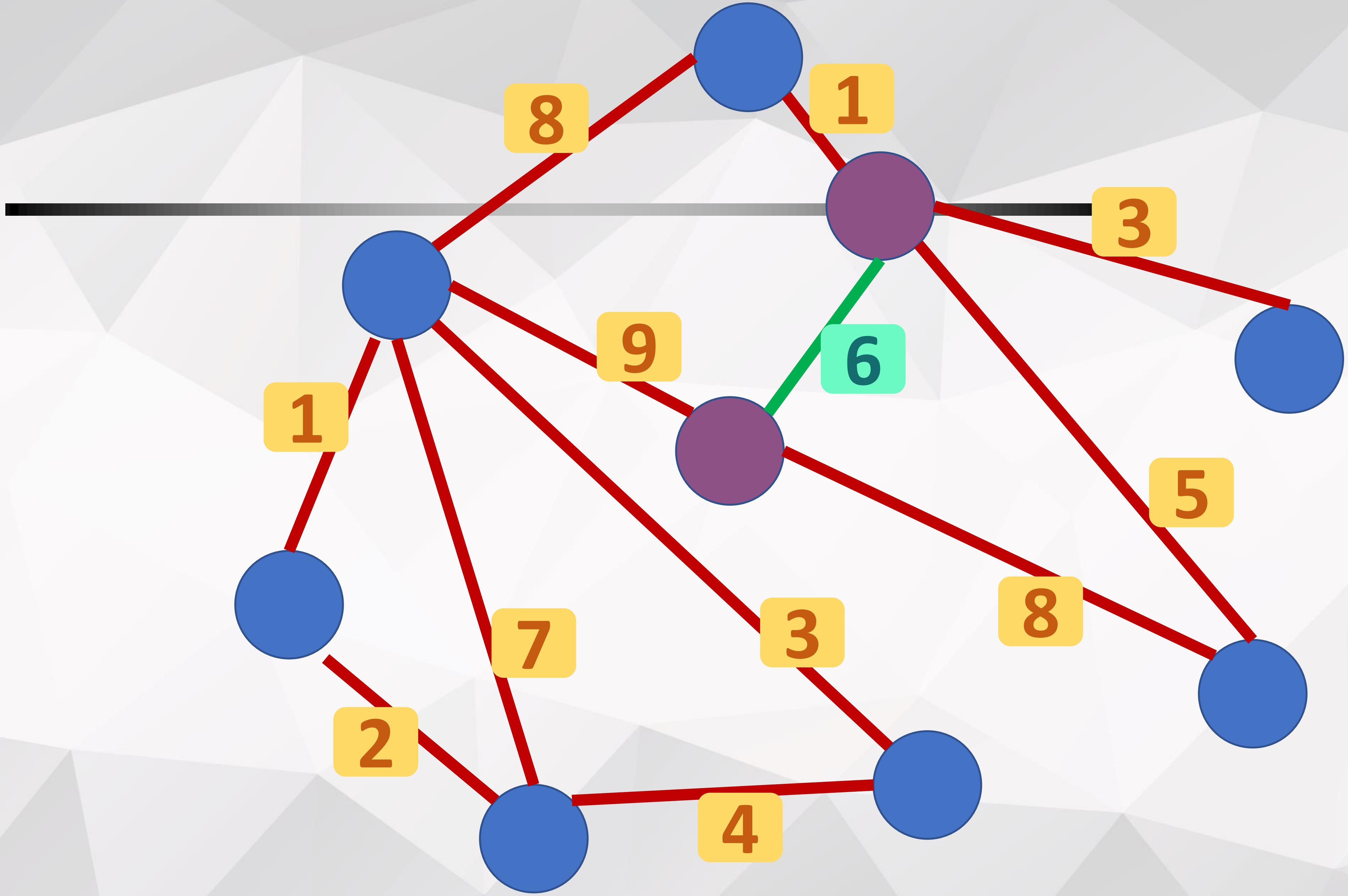


6

MST

6





6

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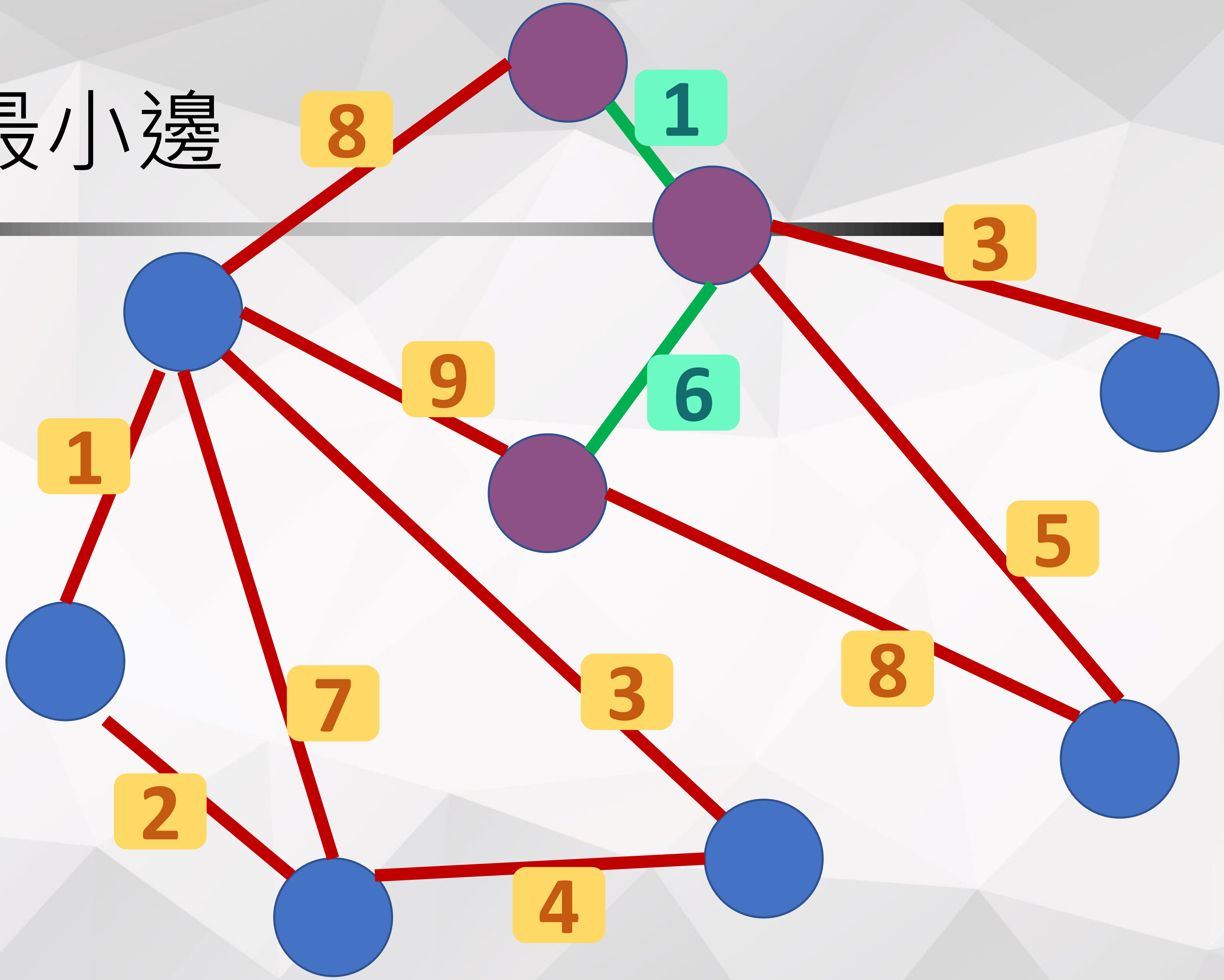
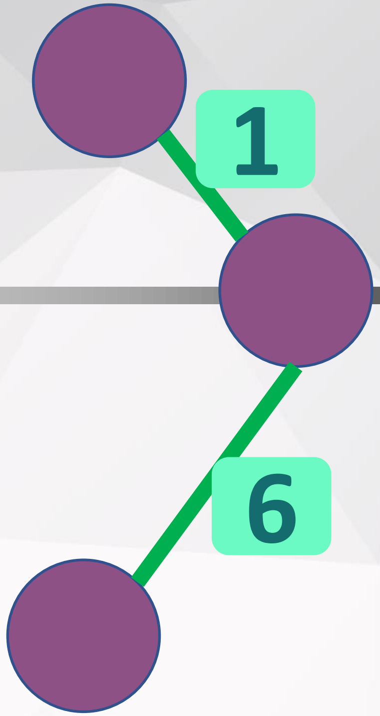
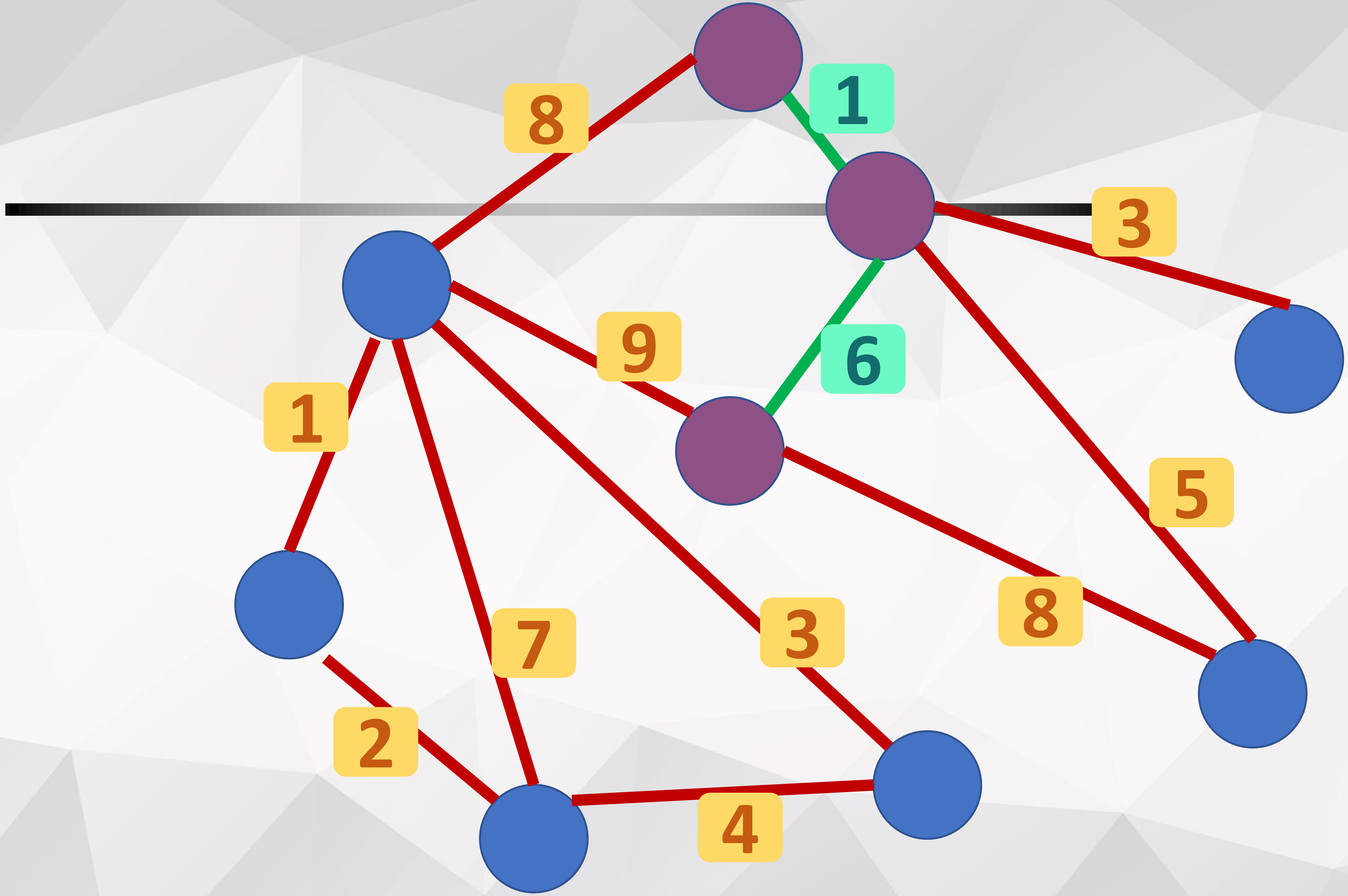


7

Competitive Programming

MST

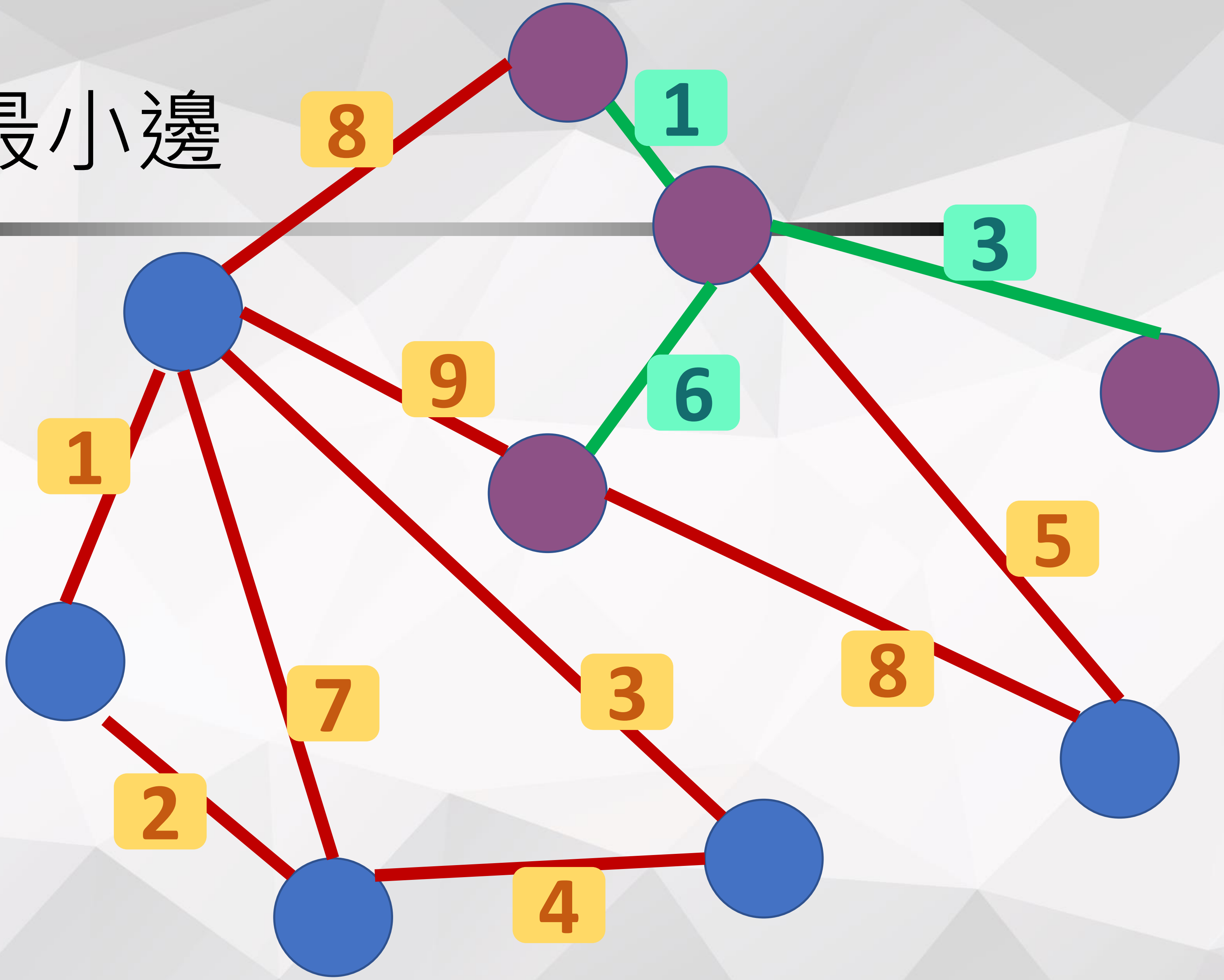




7

Competitive Programming

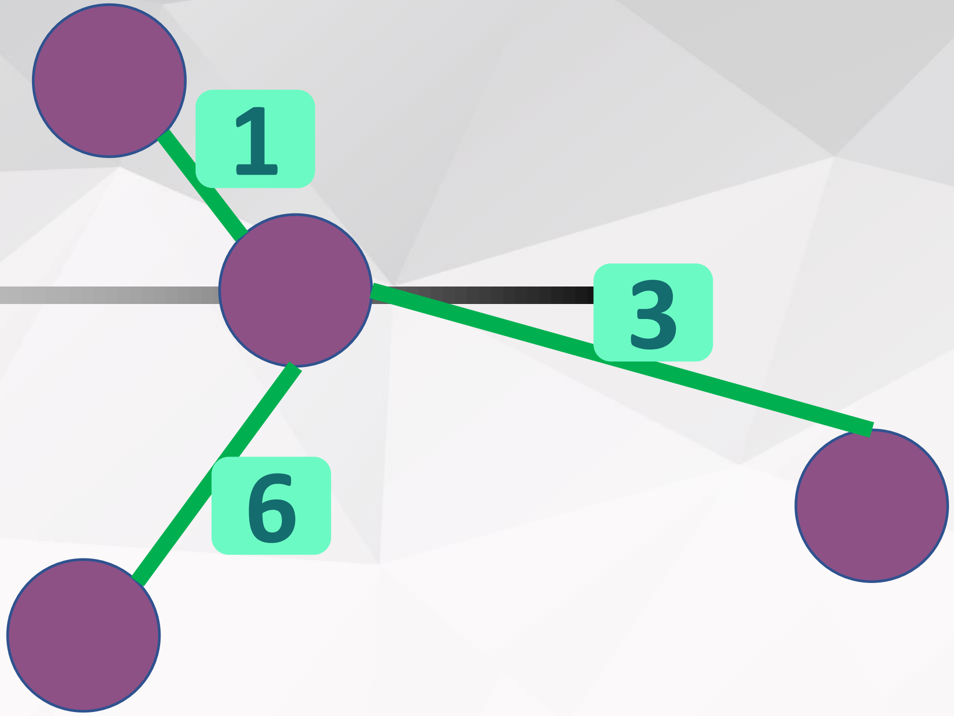
周圍最小邊

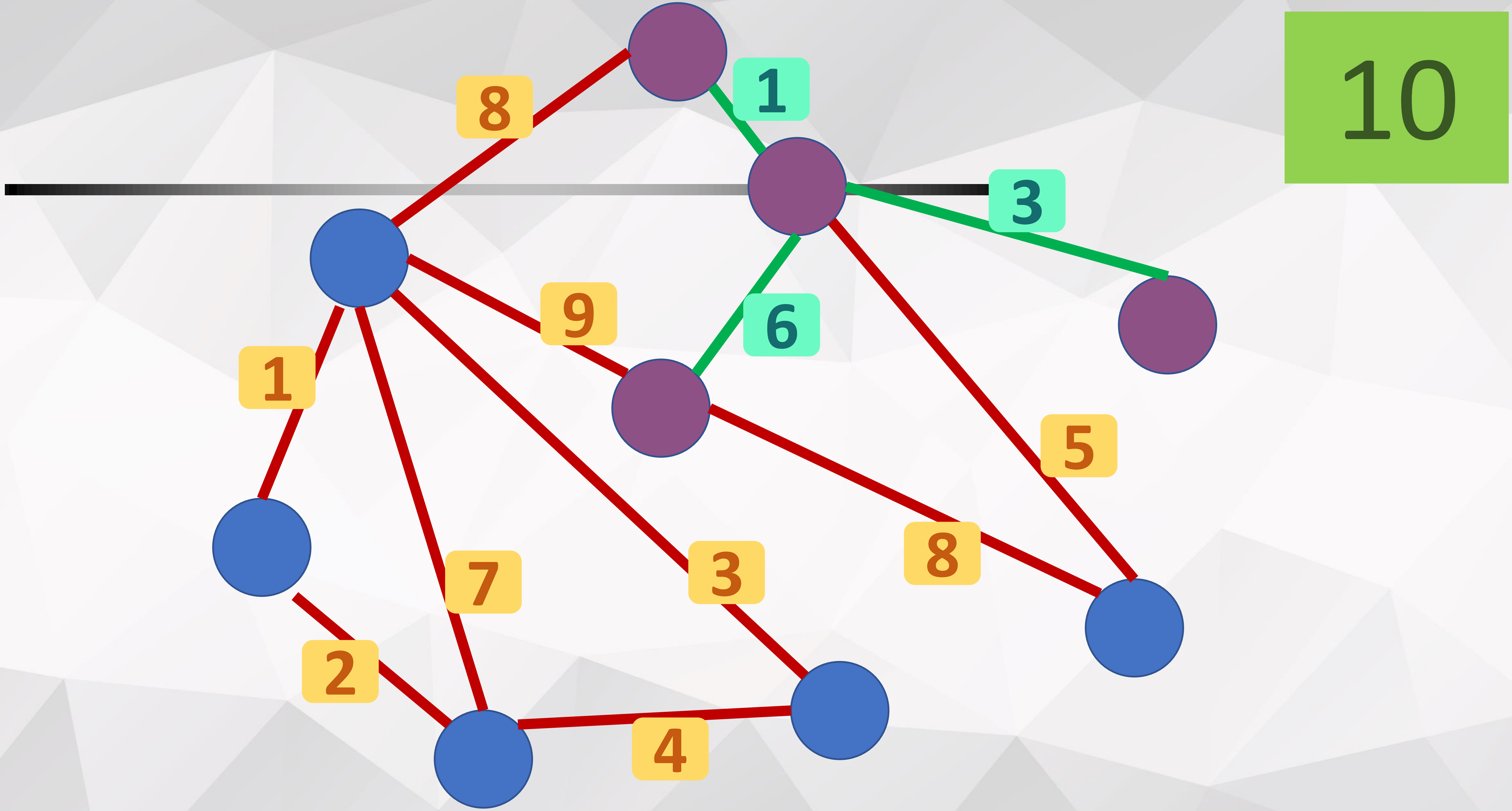


10

Competitive Programming

MST

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周圍最小邊

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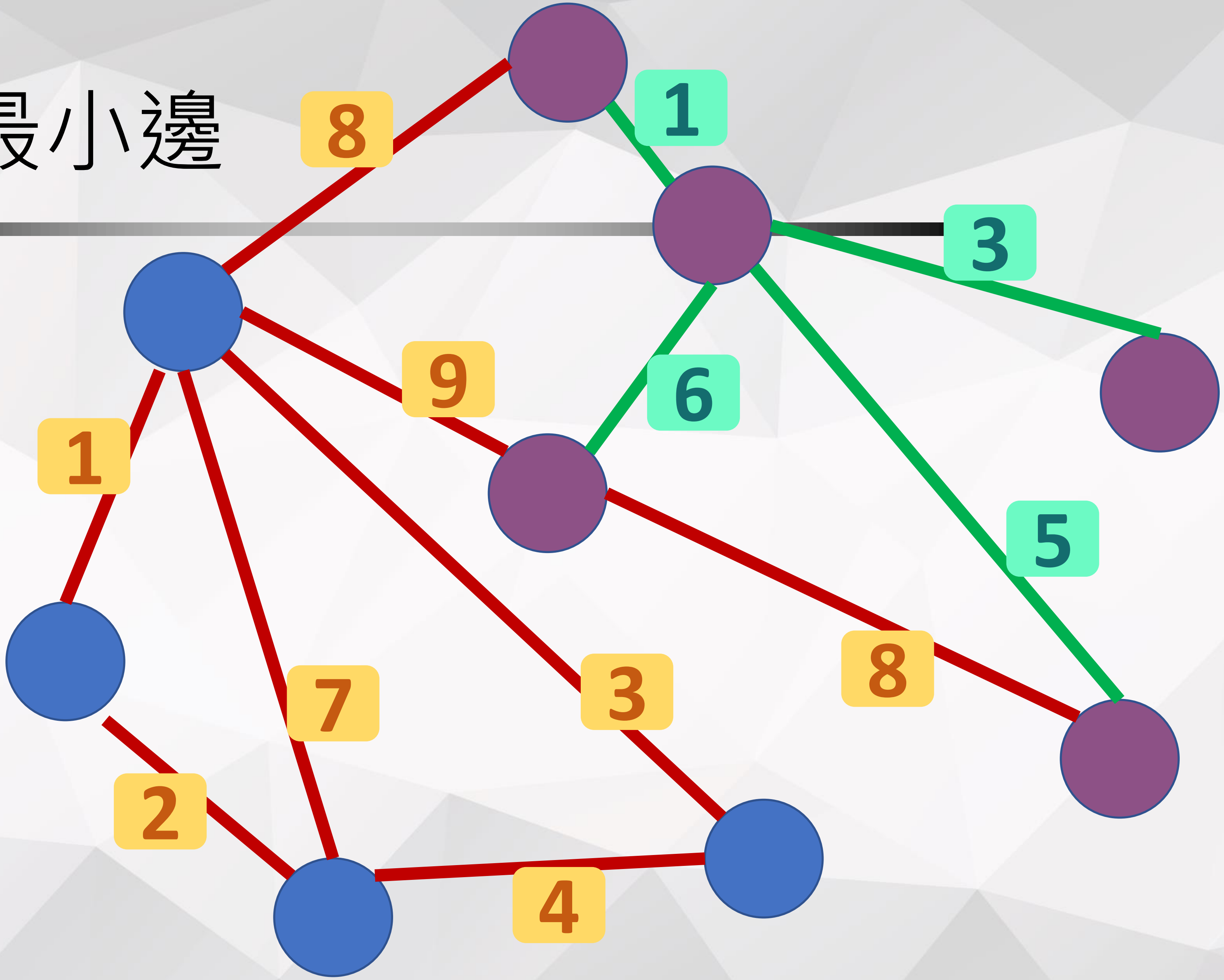

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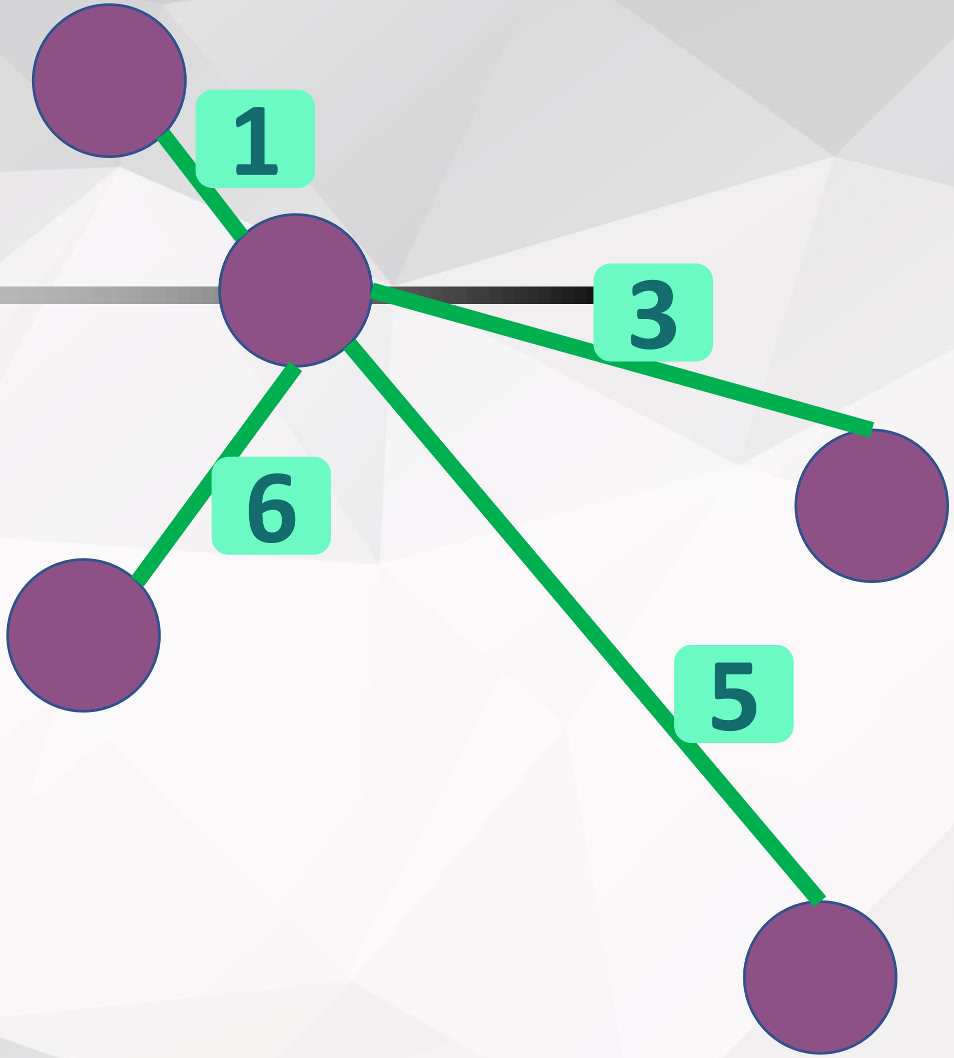
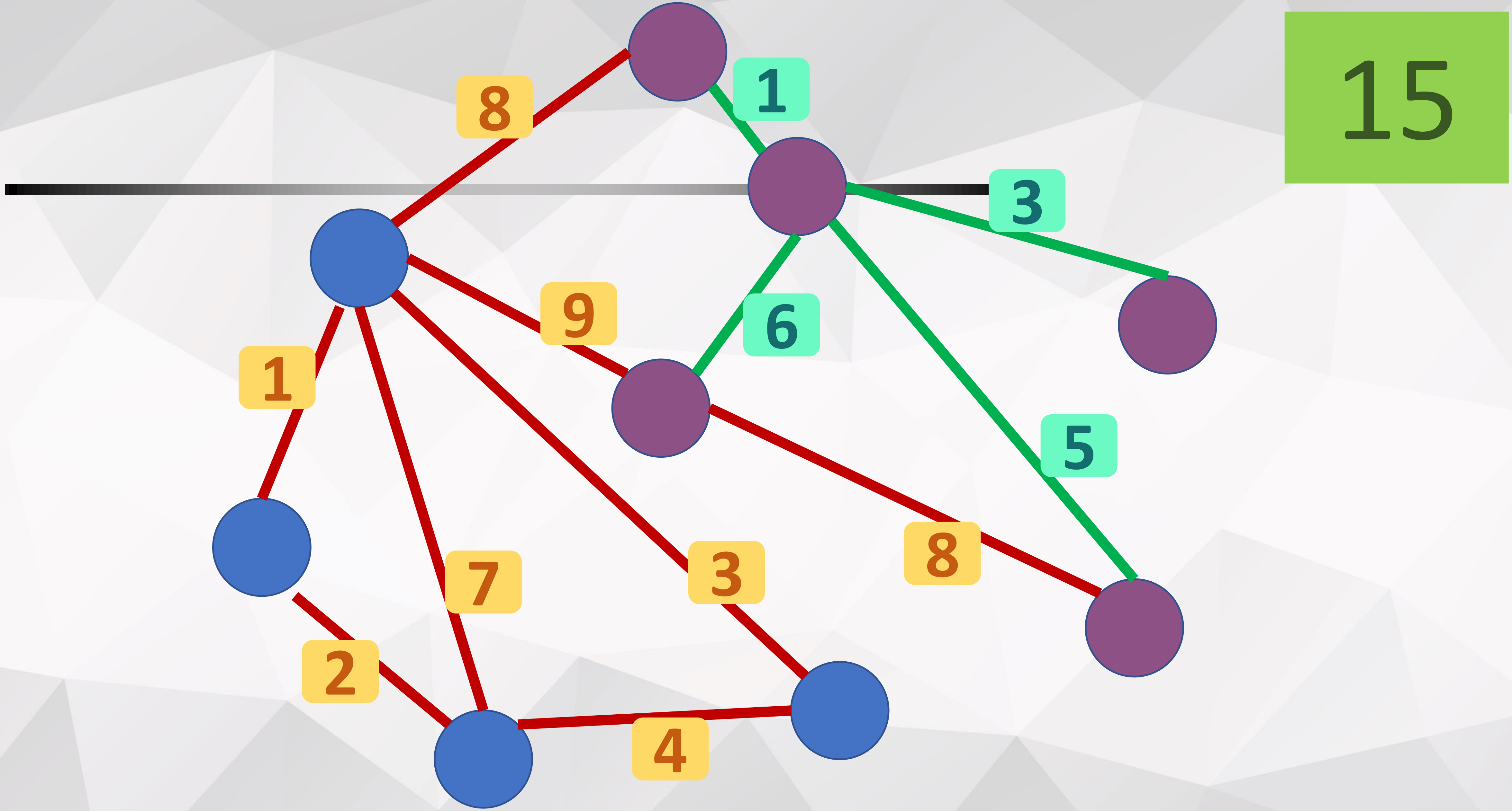
MST

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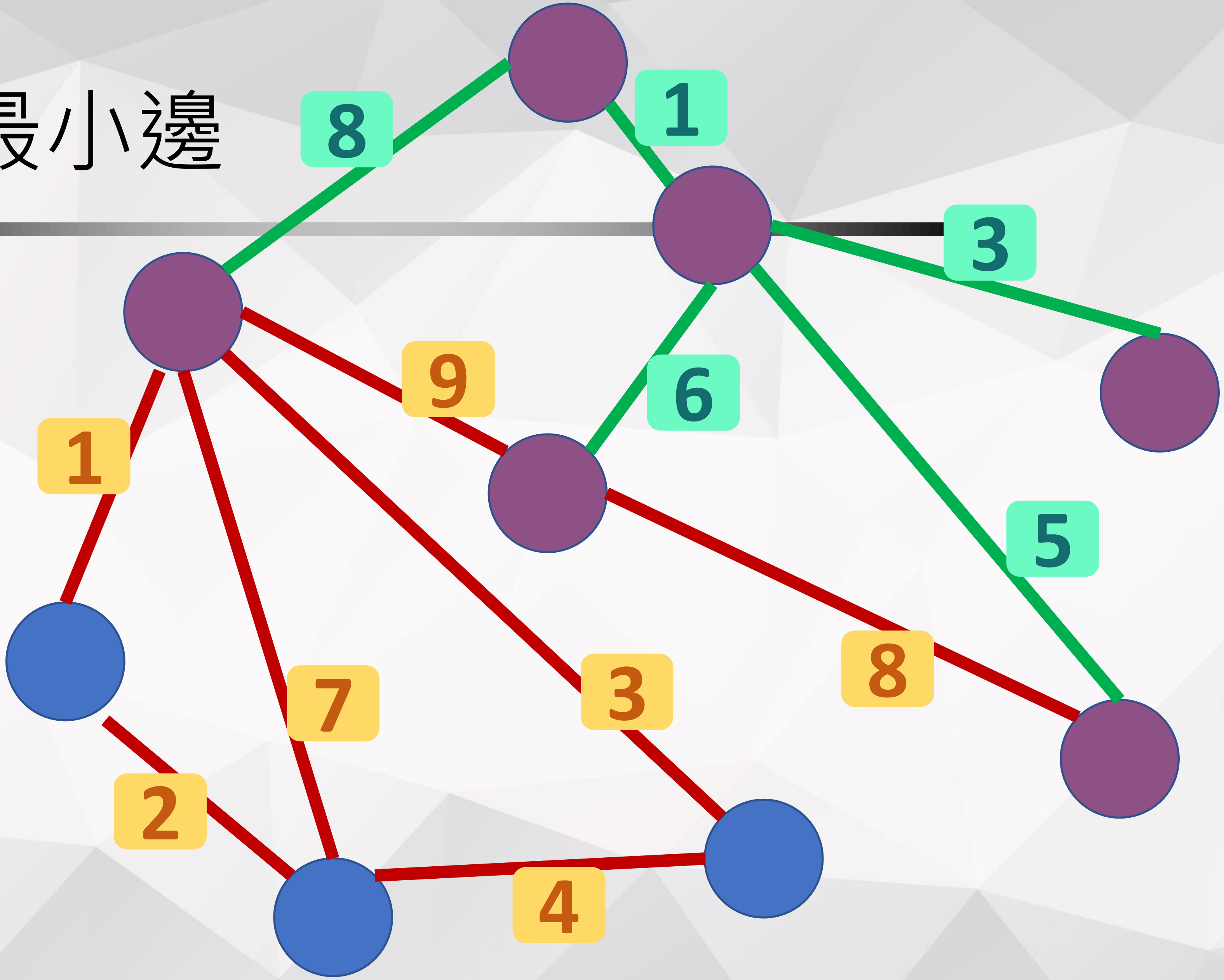
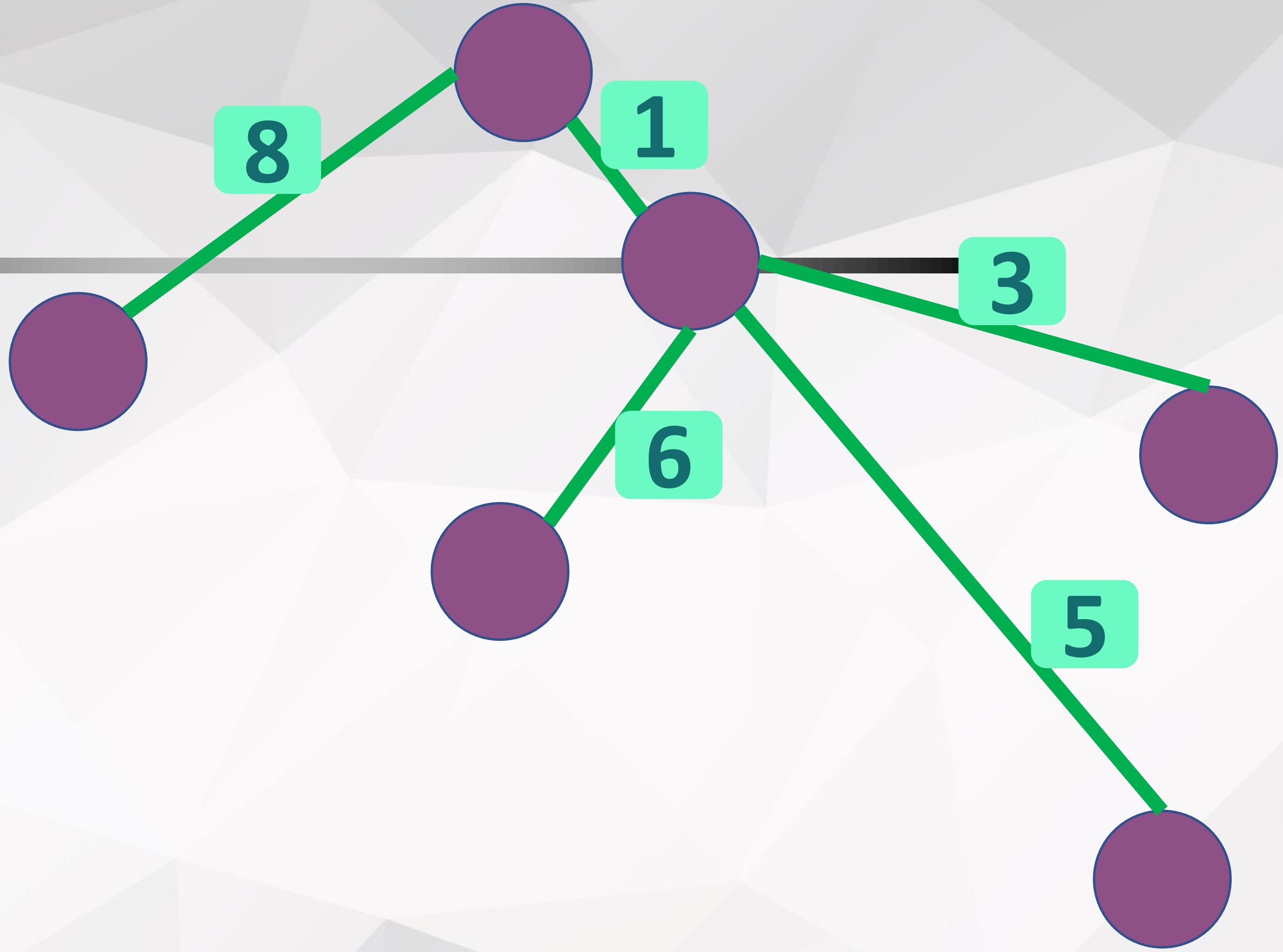
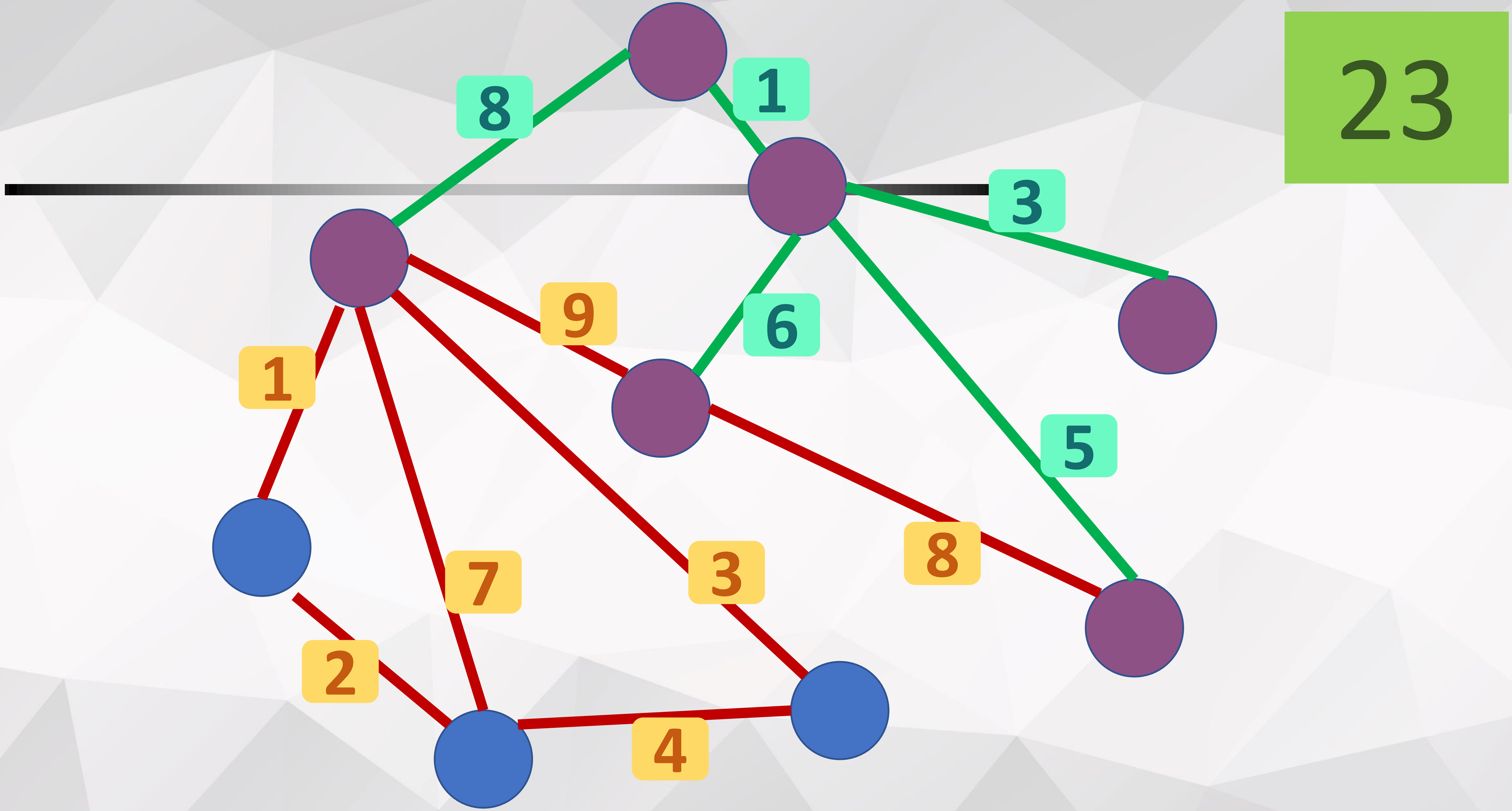
周圍最小邊



23

MST





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周圍最小邊

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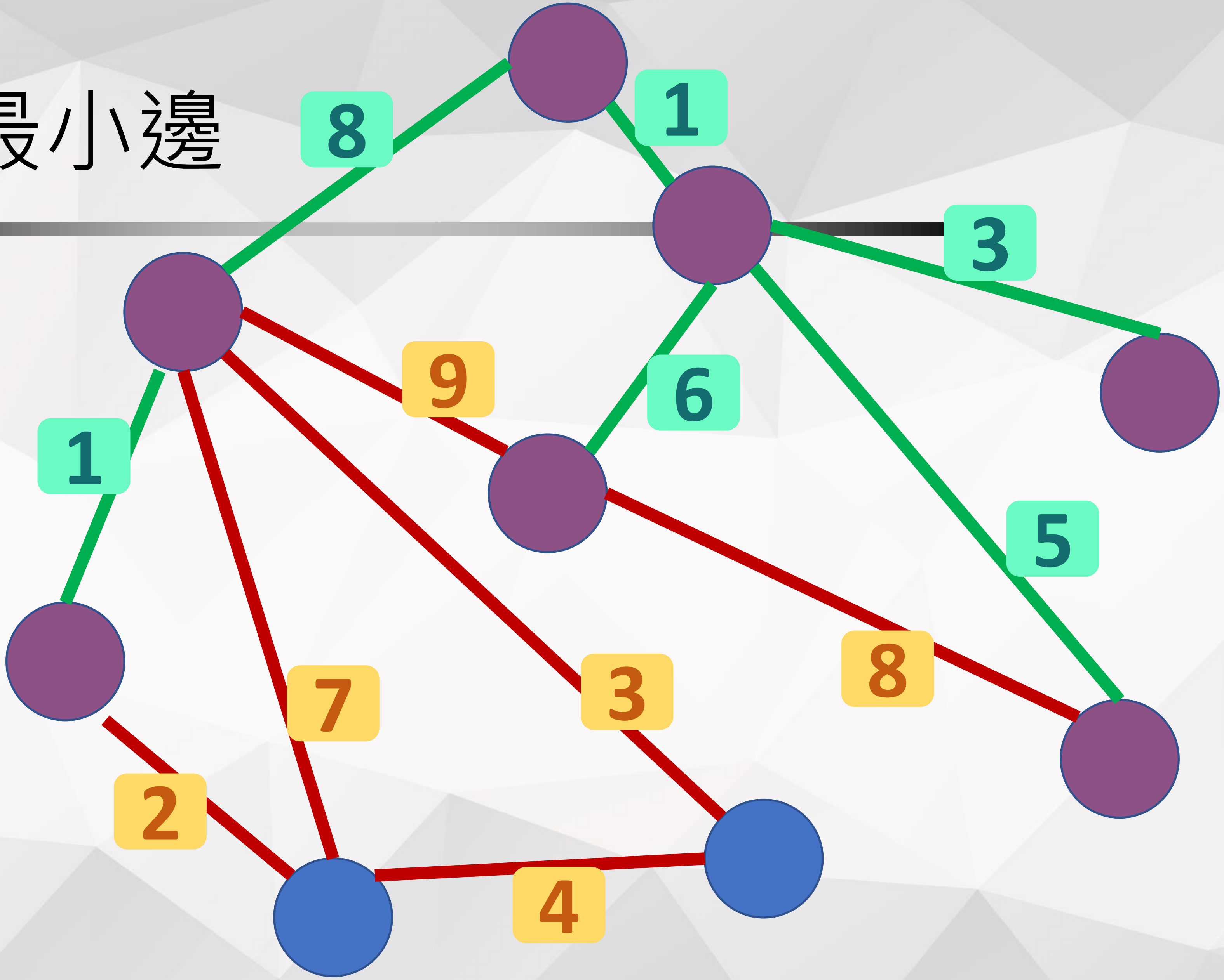

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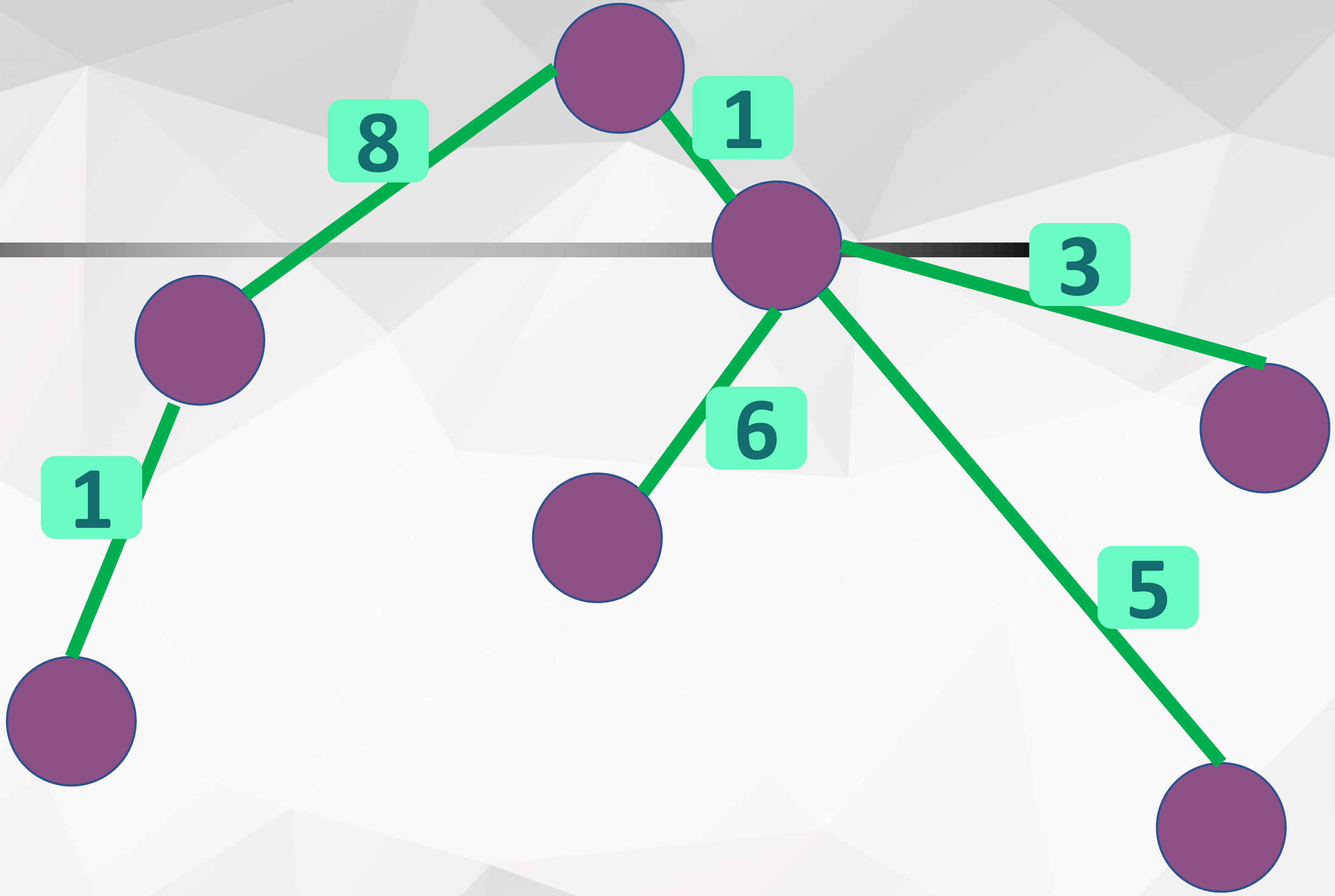

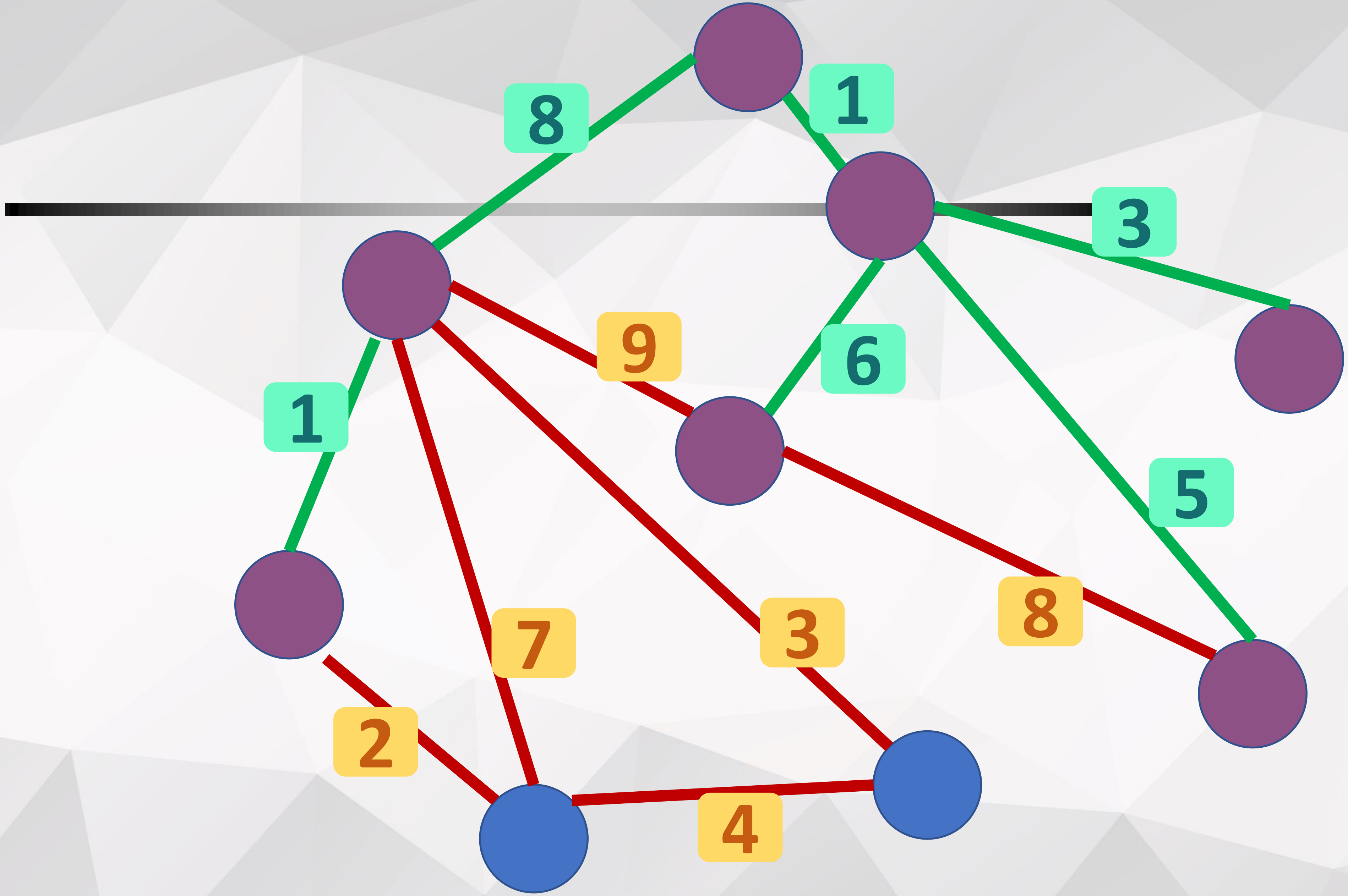
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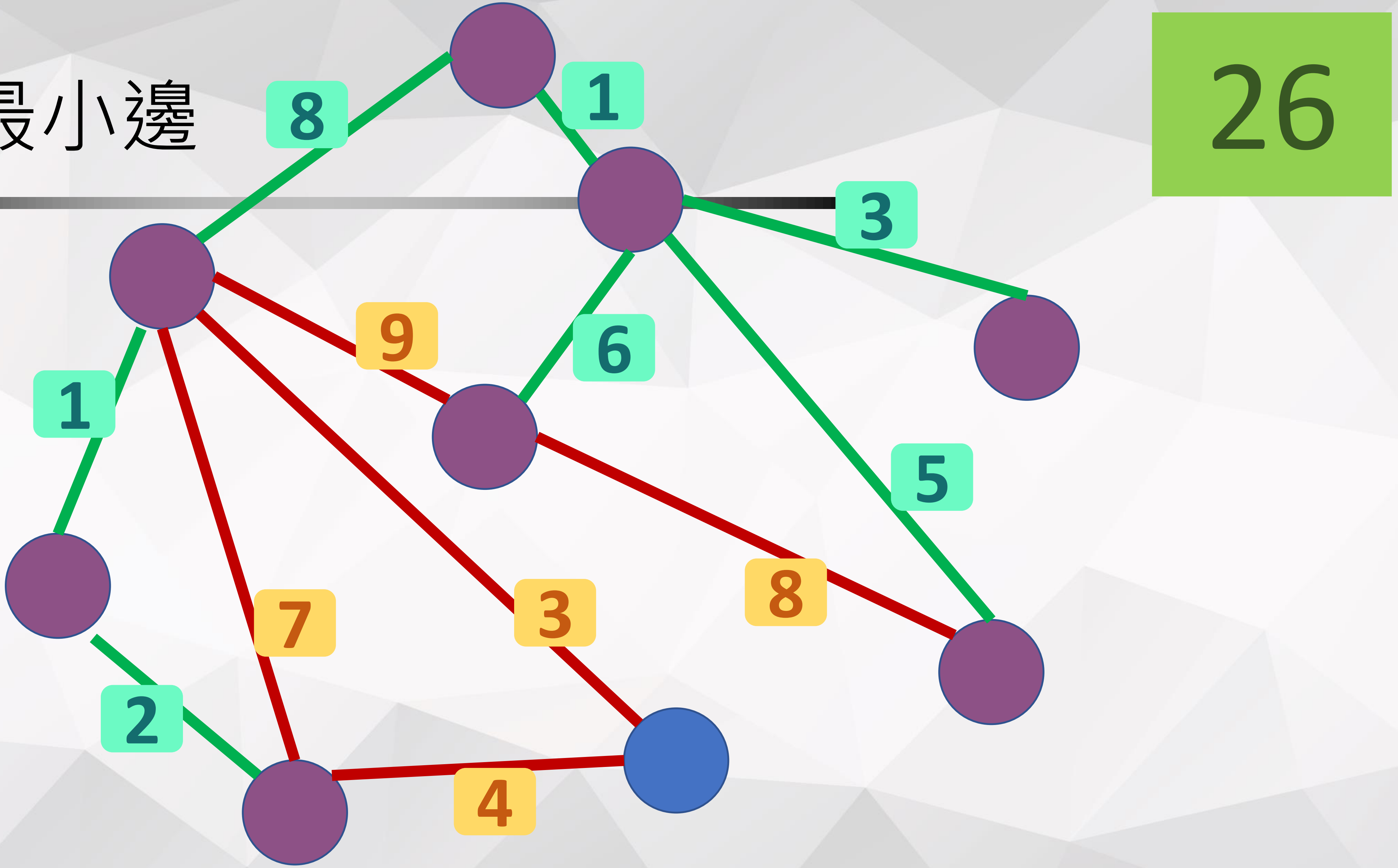
周圍最小邊



MST

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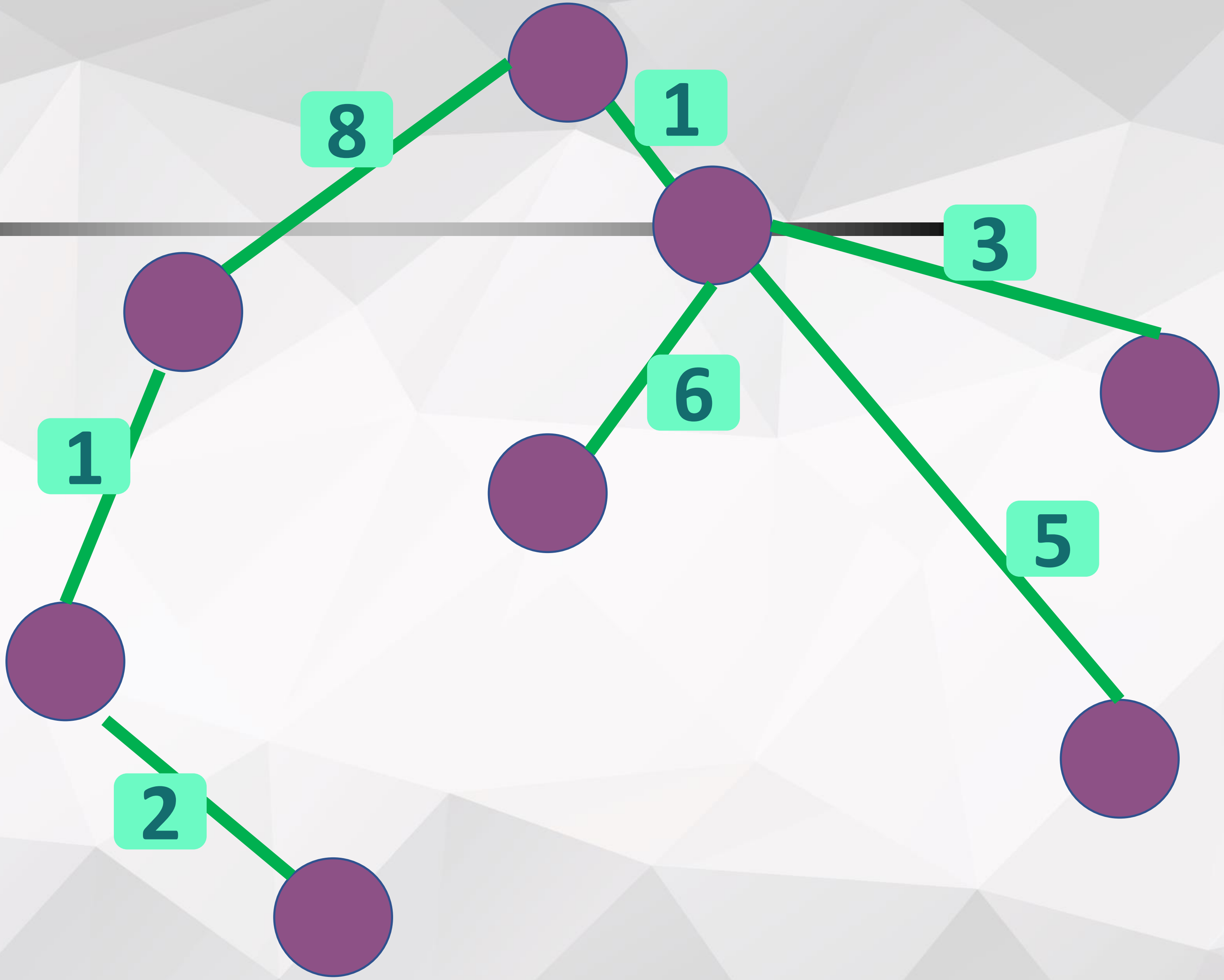

Competitive Programming

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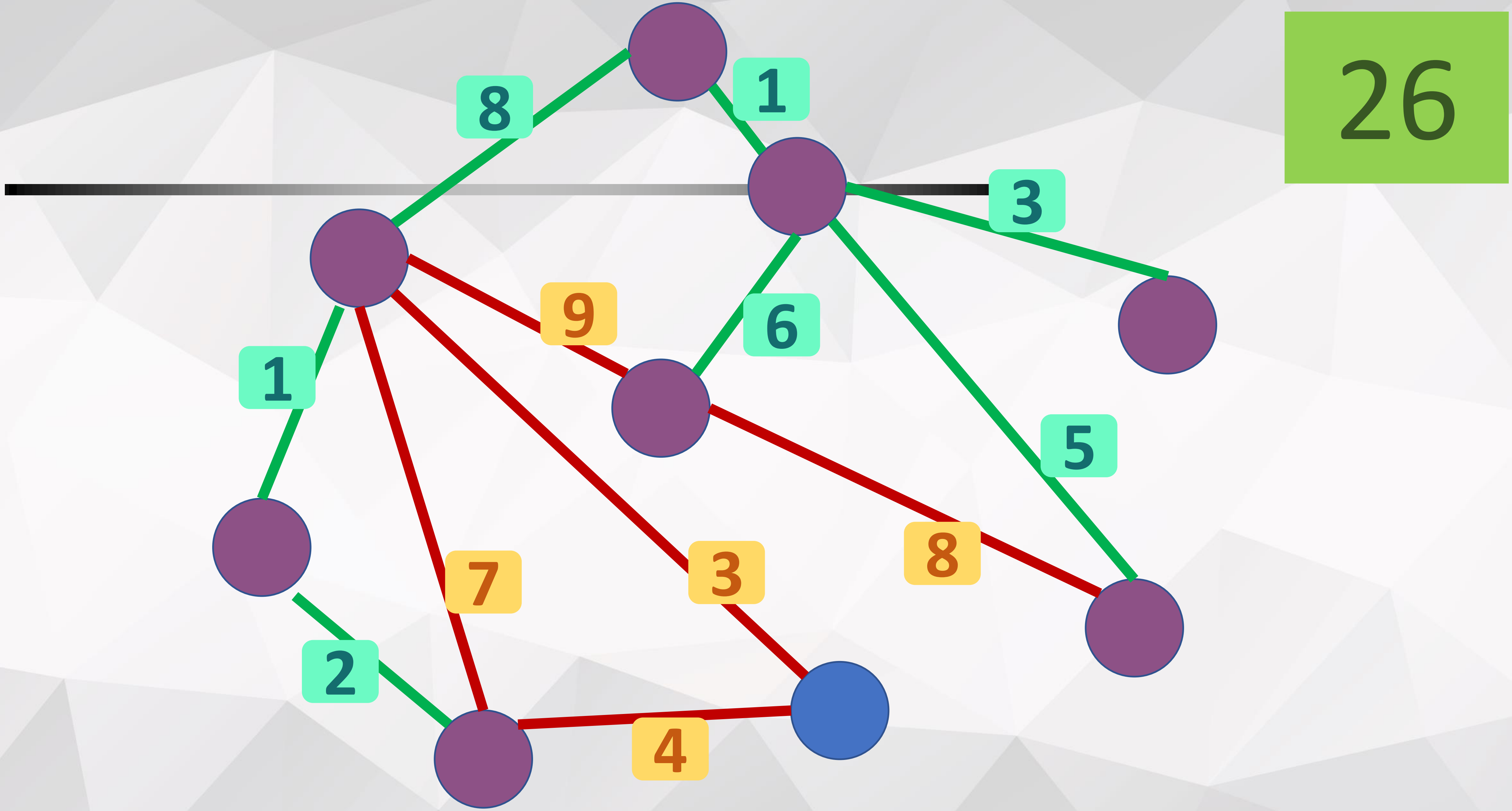

Competitive Programming

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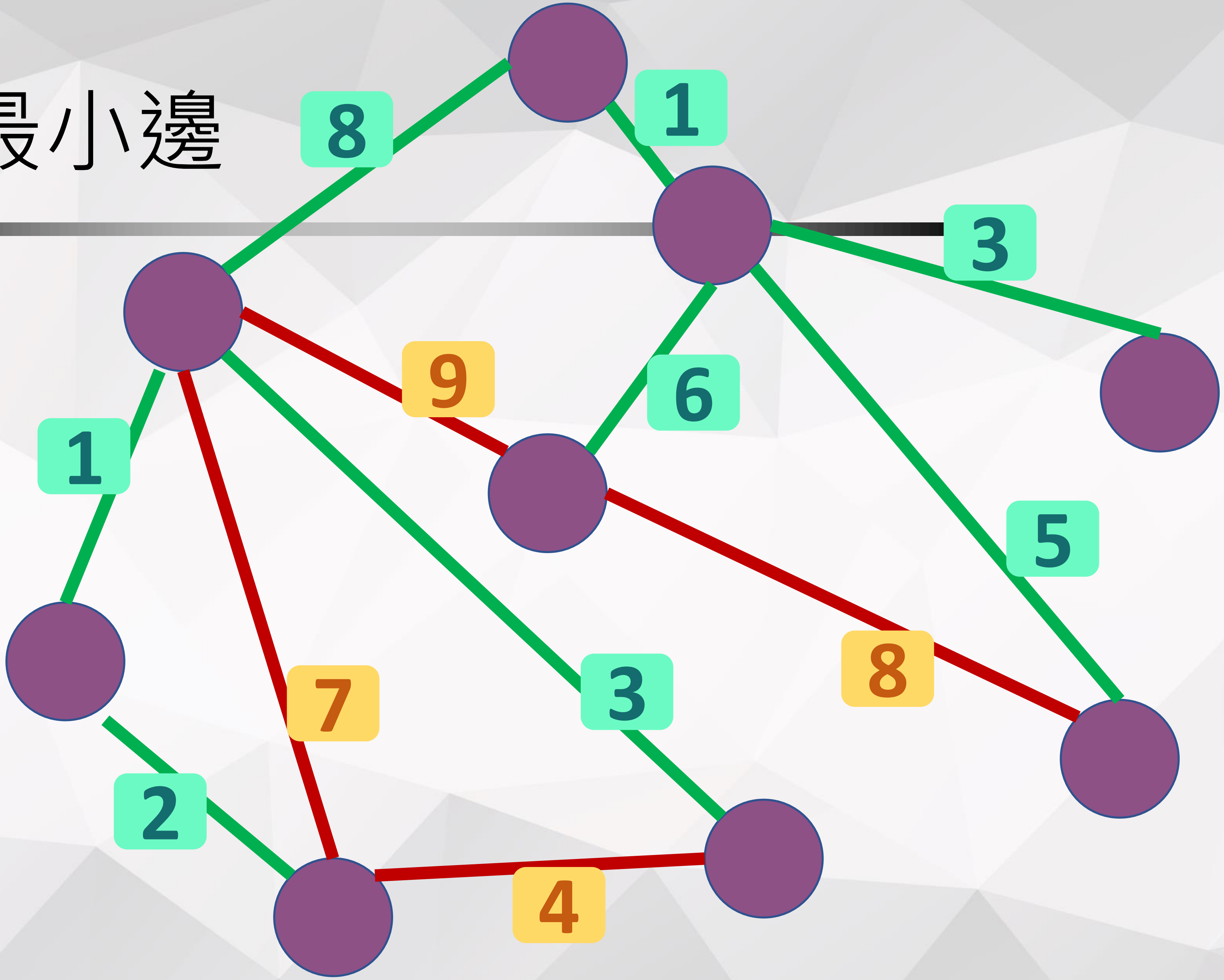
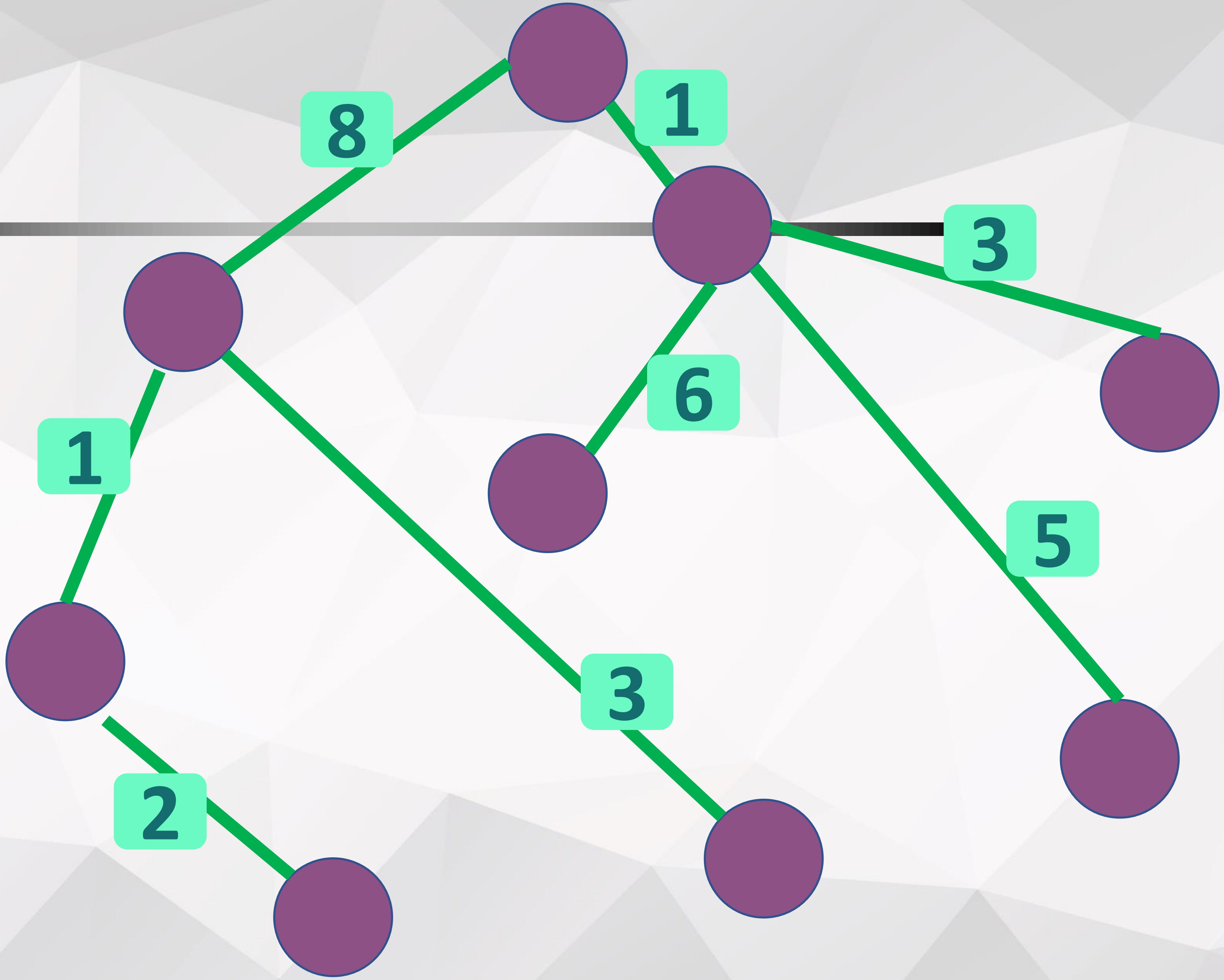
周圍最小邊



MST



Prim 實作

```
struct node {
   int id; // 點的編號
   int w; // 連結到此點的權重 (邊權重)
};
```

Prim 實作

```
vector<node> E[maxv]; // maxv 為最大節點數
    :
    .

/* 假設輸入完邊的資訊了 */
```

Prim 實作

```
/* 每次挑最小權重的邊 */
priority_queue<edge> Q;



/* 初始的生成樹 (只有一個點) */
Q.push({1, 1, 0});
```

Prim 實作

```
while(!Q.empty()) {
   edge e = Q.top(); Q.pop();
   int u = e.v;

   if(!vis[u]) { // 避免出現環
      MST.push_back(e);
      cost += e.w;

      for(auto v: E[u])
         if(!vis[v.id]) Q.push({u, v.id, v.w});
   }

   vis[u] = true;
}
```

Prim 實作

跟 Kruskal 比較

Prim 枚舉的是**點**
Kruskal 枚舉的是**邊**

其複雜度為 $O(|E|\log_2|V|)$

Questions?

Outline

- 術語複習
  - Graph
  - Tree
- 最小生成樹
- A* 搜尋法則
- 單源最短路徑
- 全點對最短路徑

Single-Source Shortest Paths

SSSP

- 給定 **源**點/起點(Source)

- 問每條路徑的最小成本
  - 源點到各點的最小總成本

SSSP

- Breadth-First Search
- Relaxation
- Dijkstra's algorithm
- Bellman-Ford's algorithm

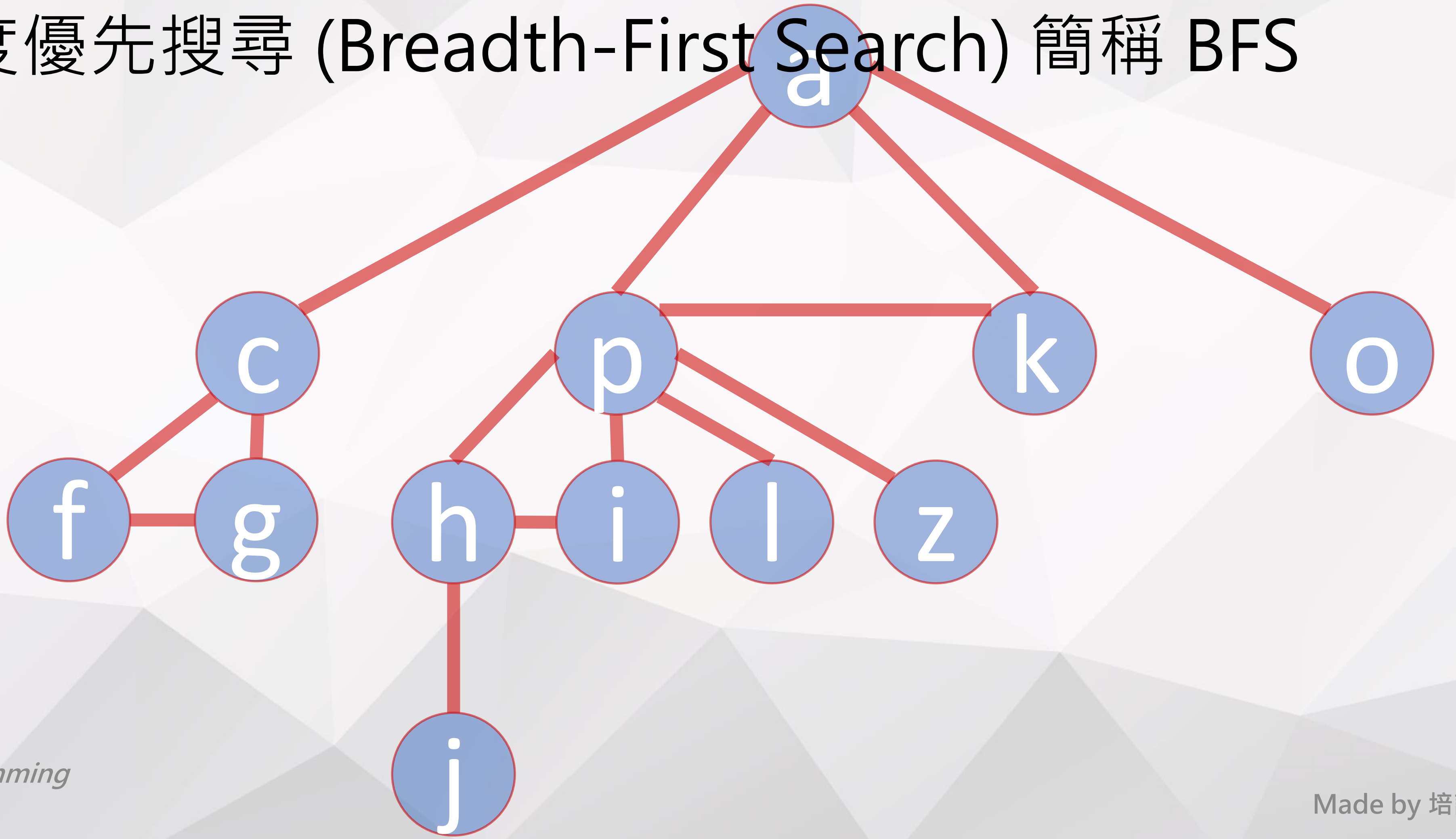
SSSP

- Breadth-First Search
- Relaxation
- Dijkstra's algorithm
- Bellman-Ford's algorithm

廣度優先搜尋

BFS

廣度優先搜尋 (Breadth-First Search) 簡稱 BFS

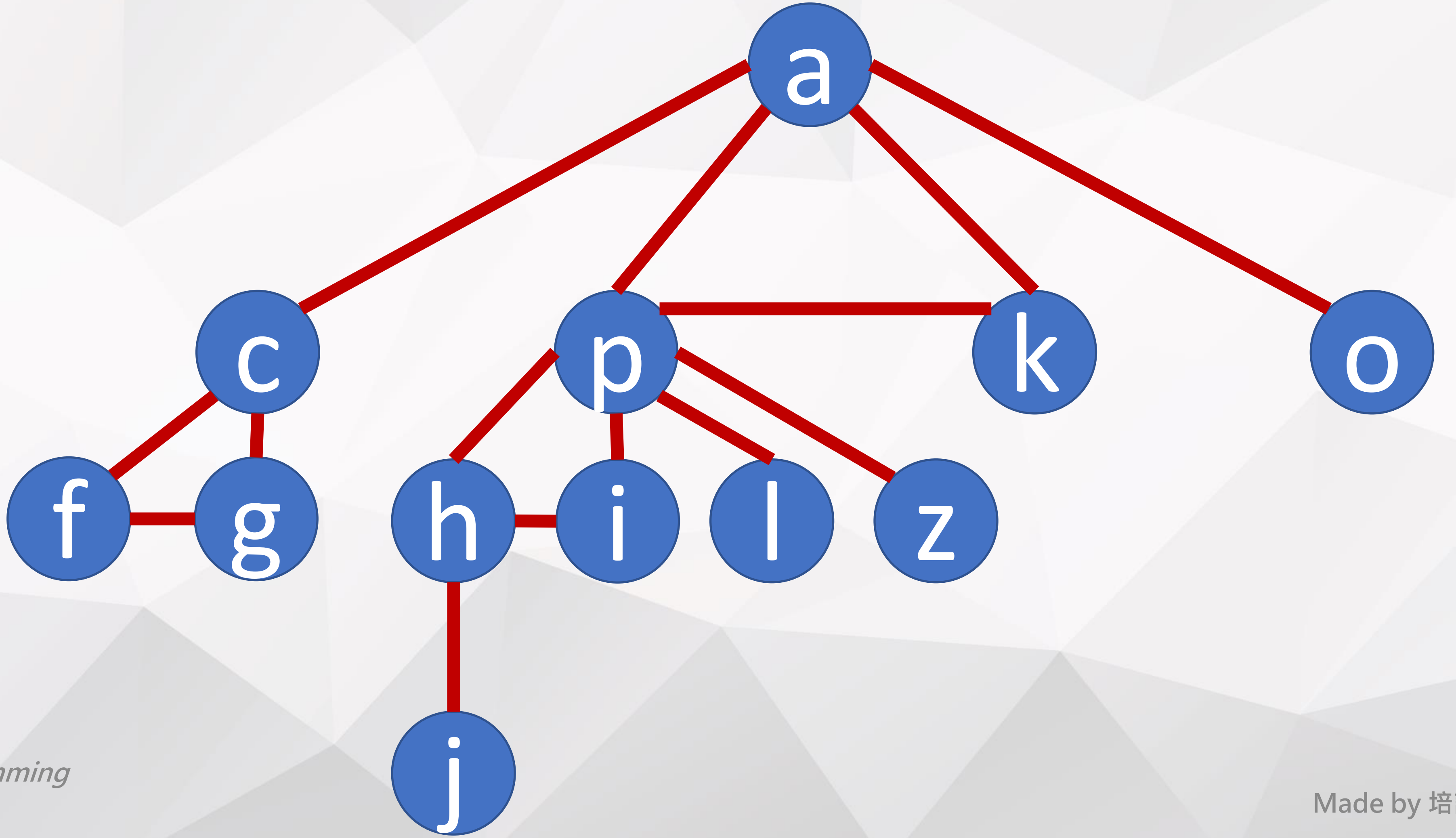


BFS 程式碼

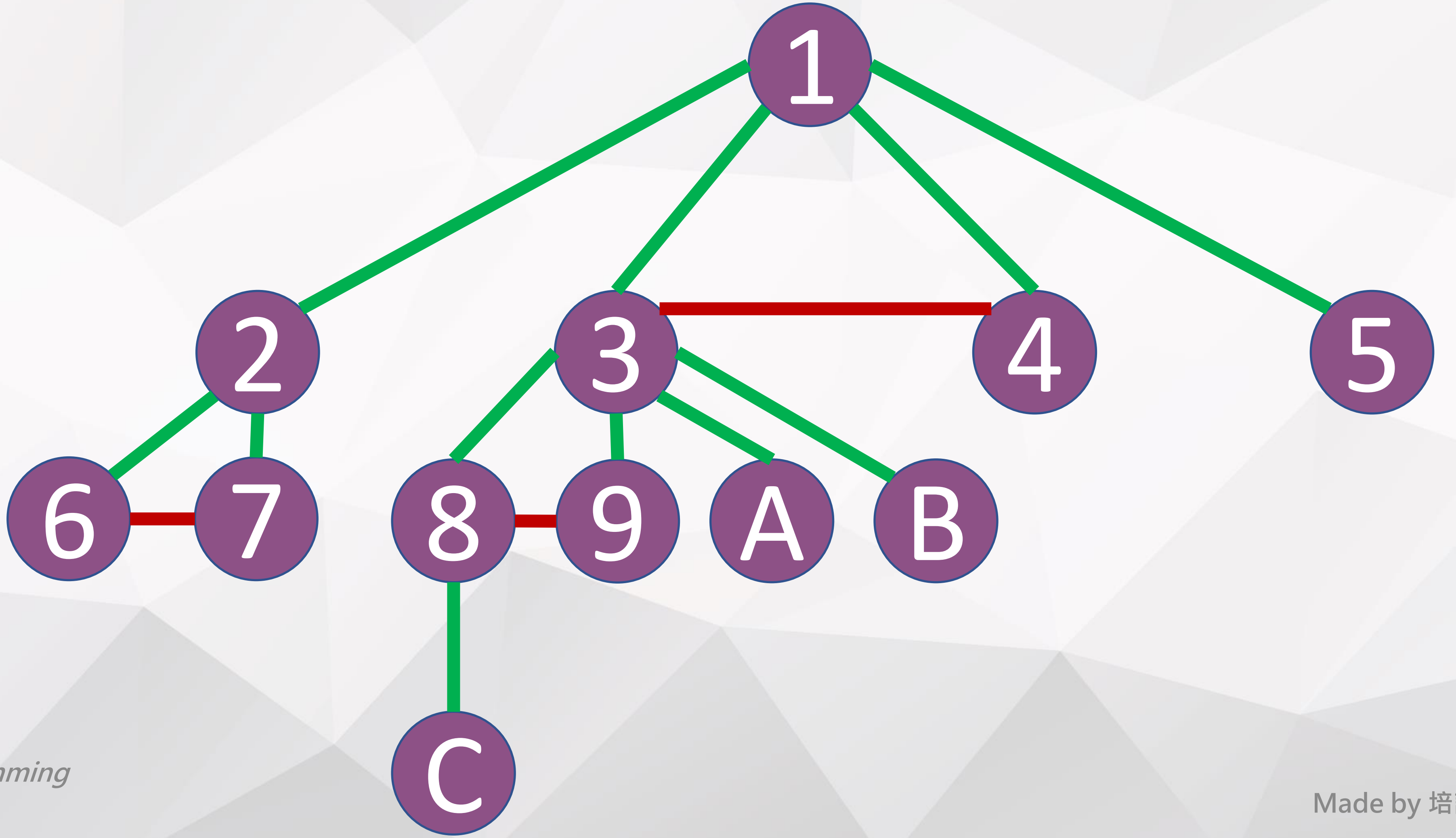
```
queue<int> Q;
Q.push(source);
vis[source] = true;

while (!Q.empty()) {
  int u = Q.front(); Q.pop();
  for (auto v: E[u]) {
    if (vis[v]) continue;
    vis[v] = true;
    Q.push(v);
  }
}
```

BFS 的點遍歷順序

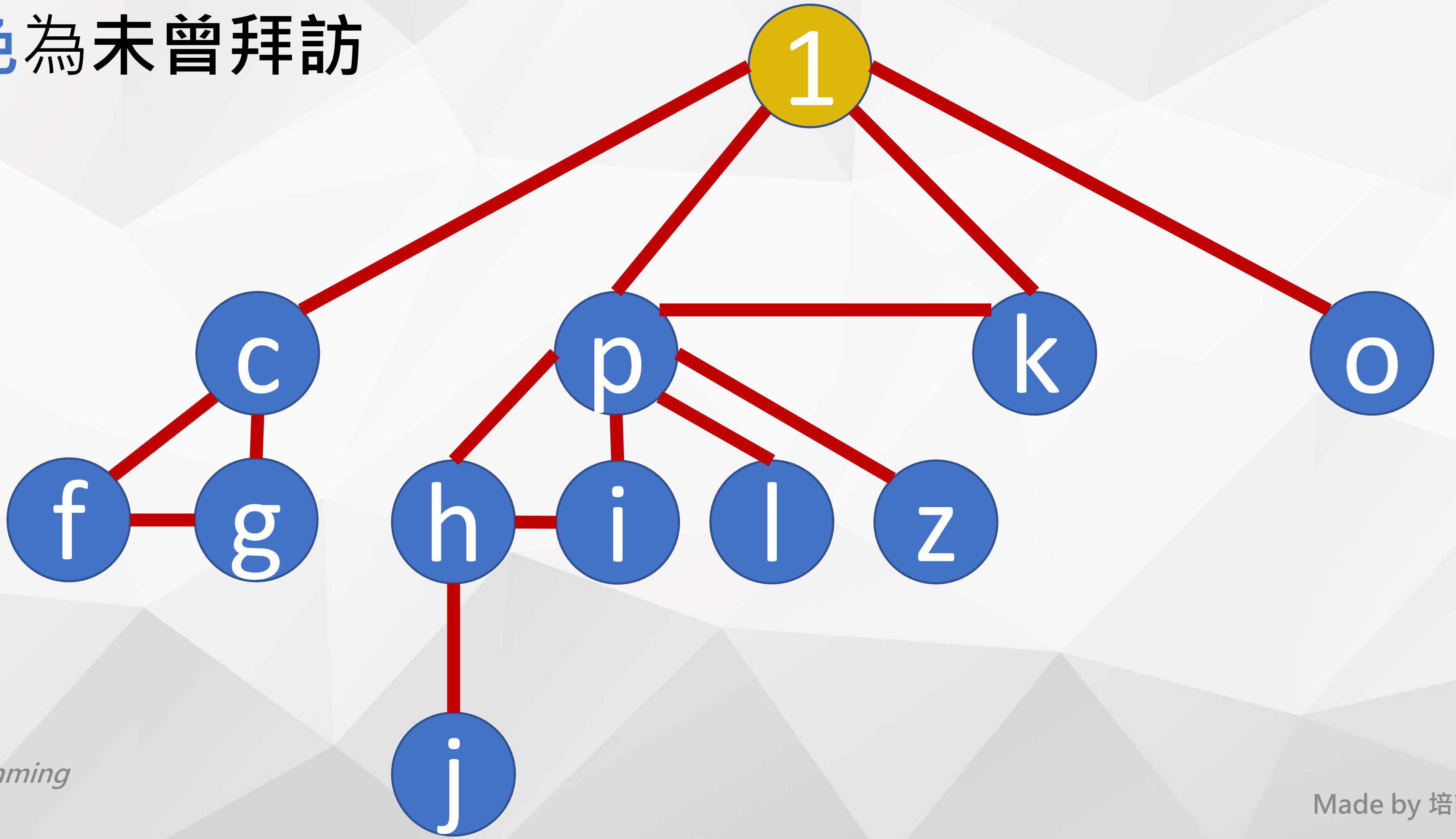


BFS 的**點**遍歷順序



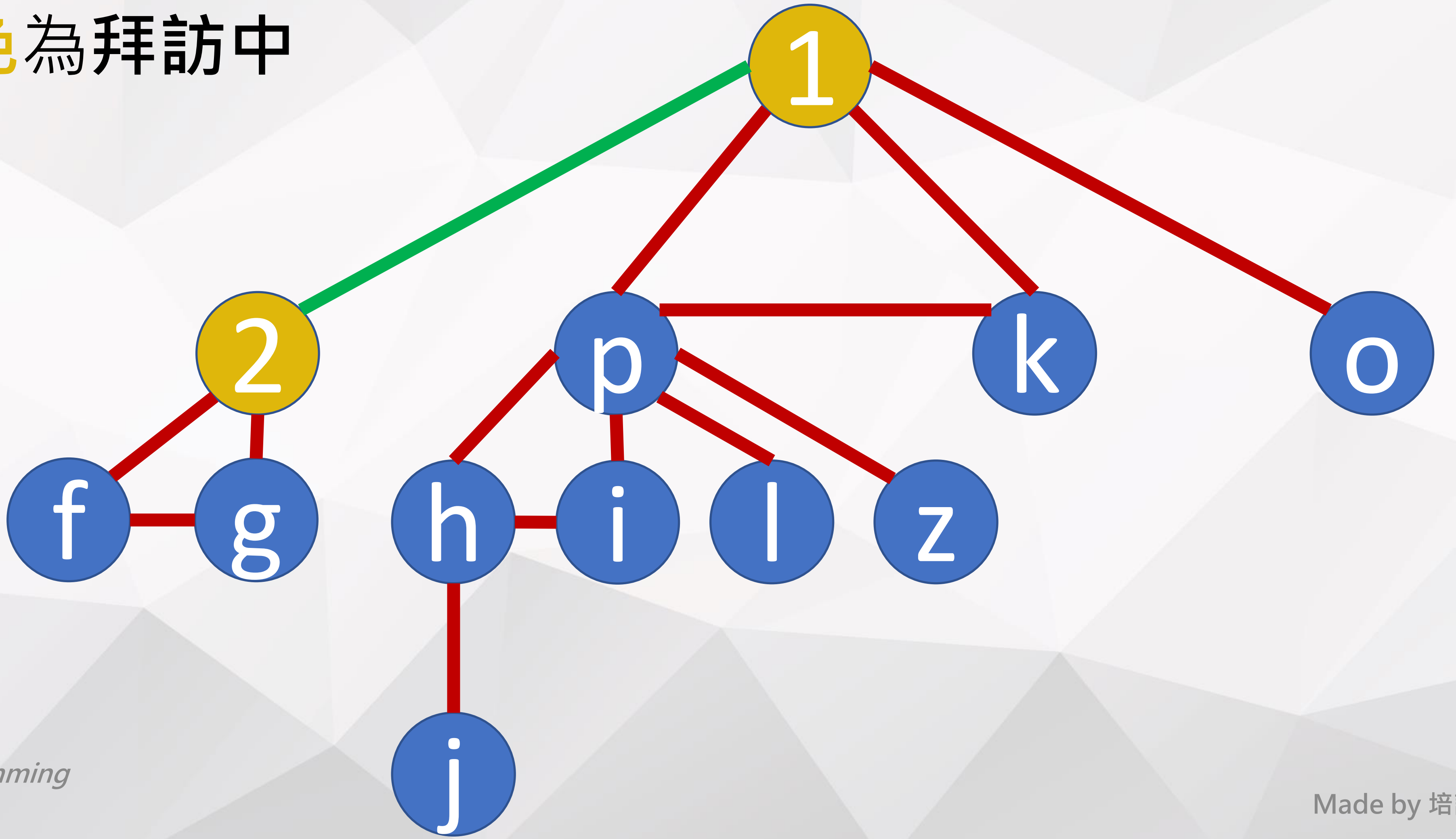
第一個拜訪的為**根**

藍色為**未曾拜訪**



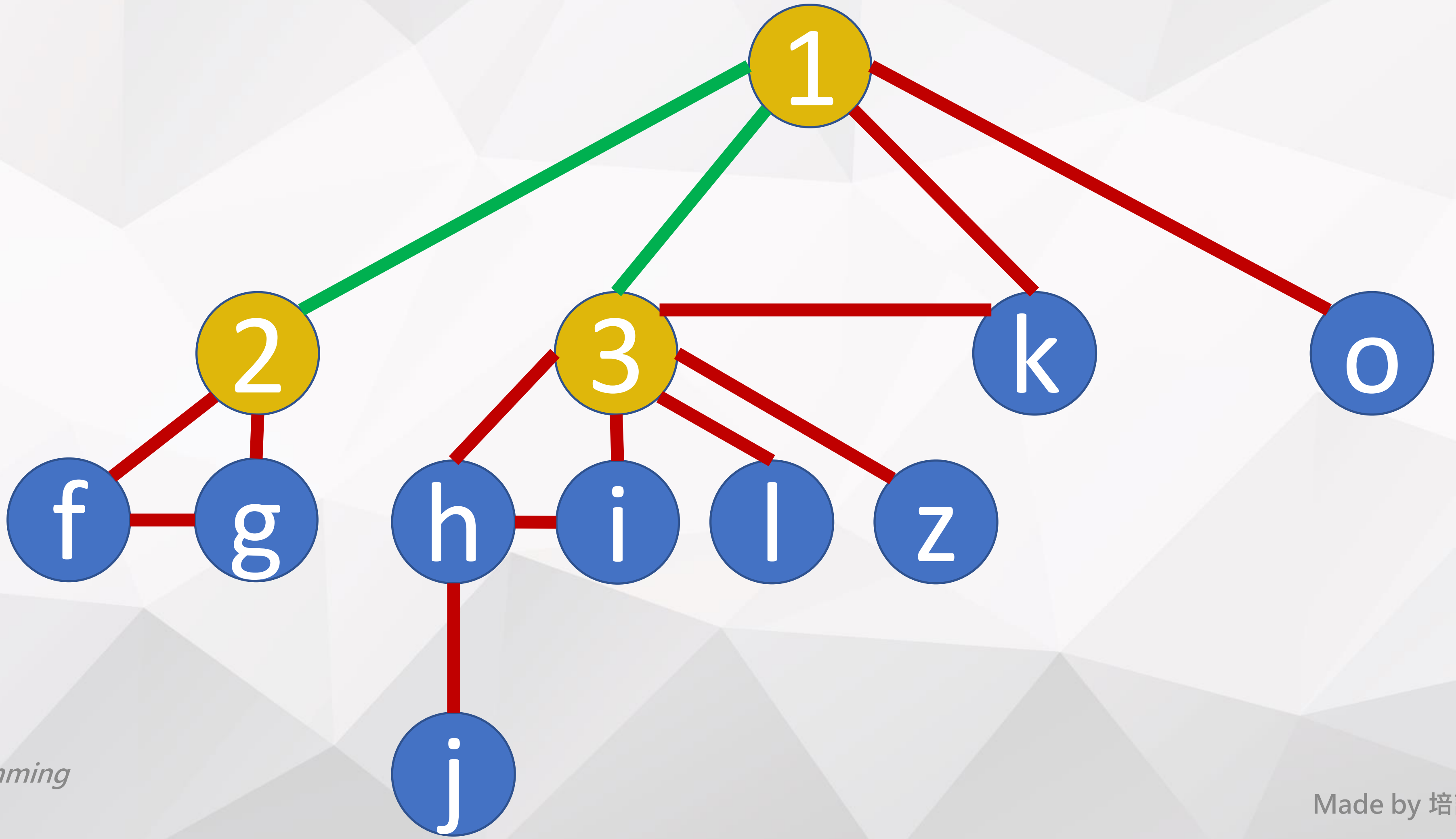
拜訪所有鄰點

黃色為拜訪中



Competitive Programming

拜訪**所有**鄰點



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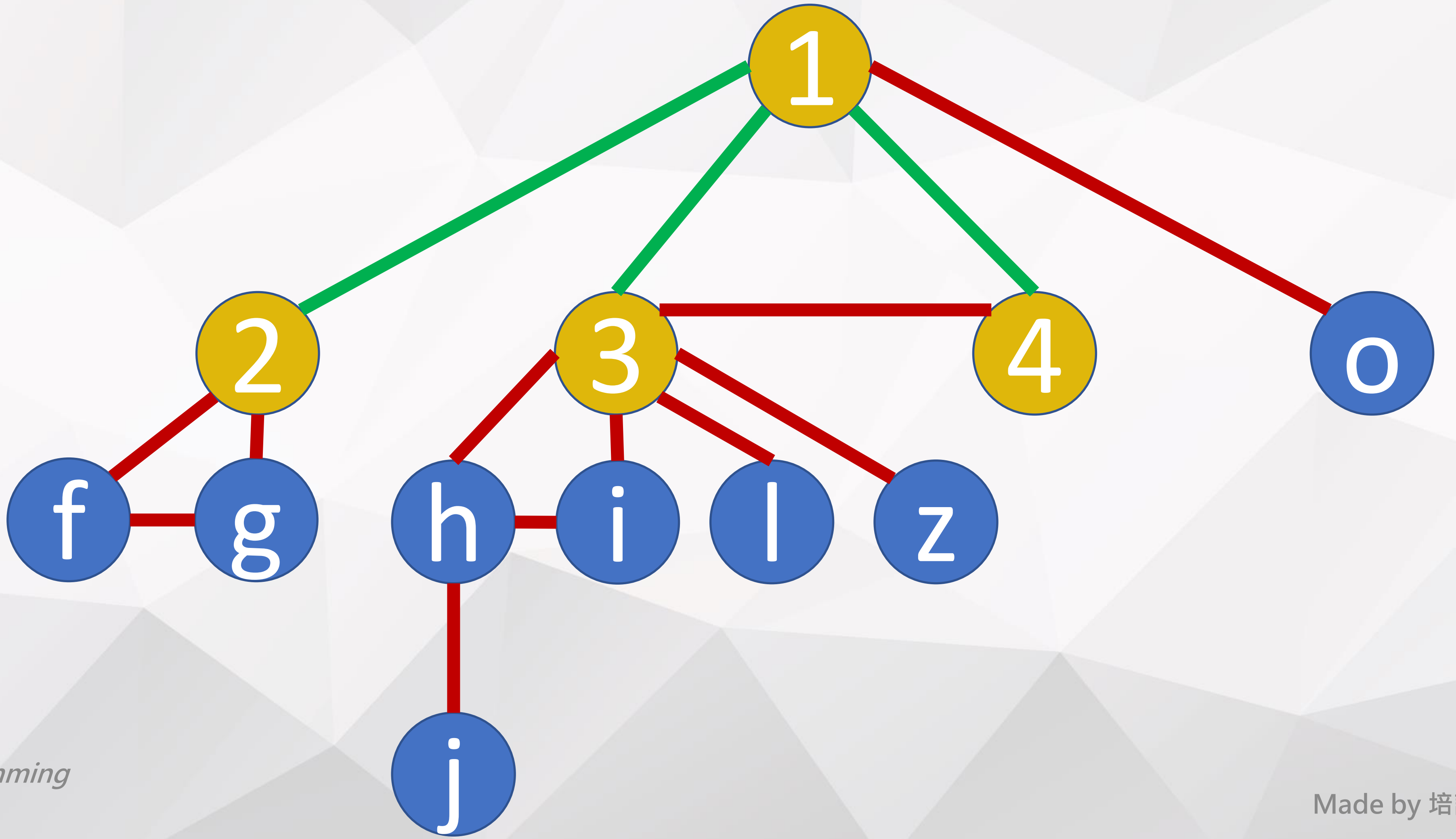

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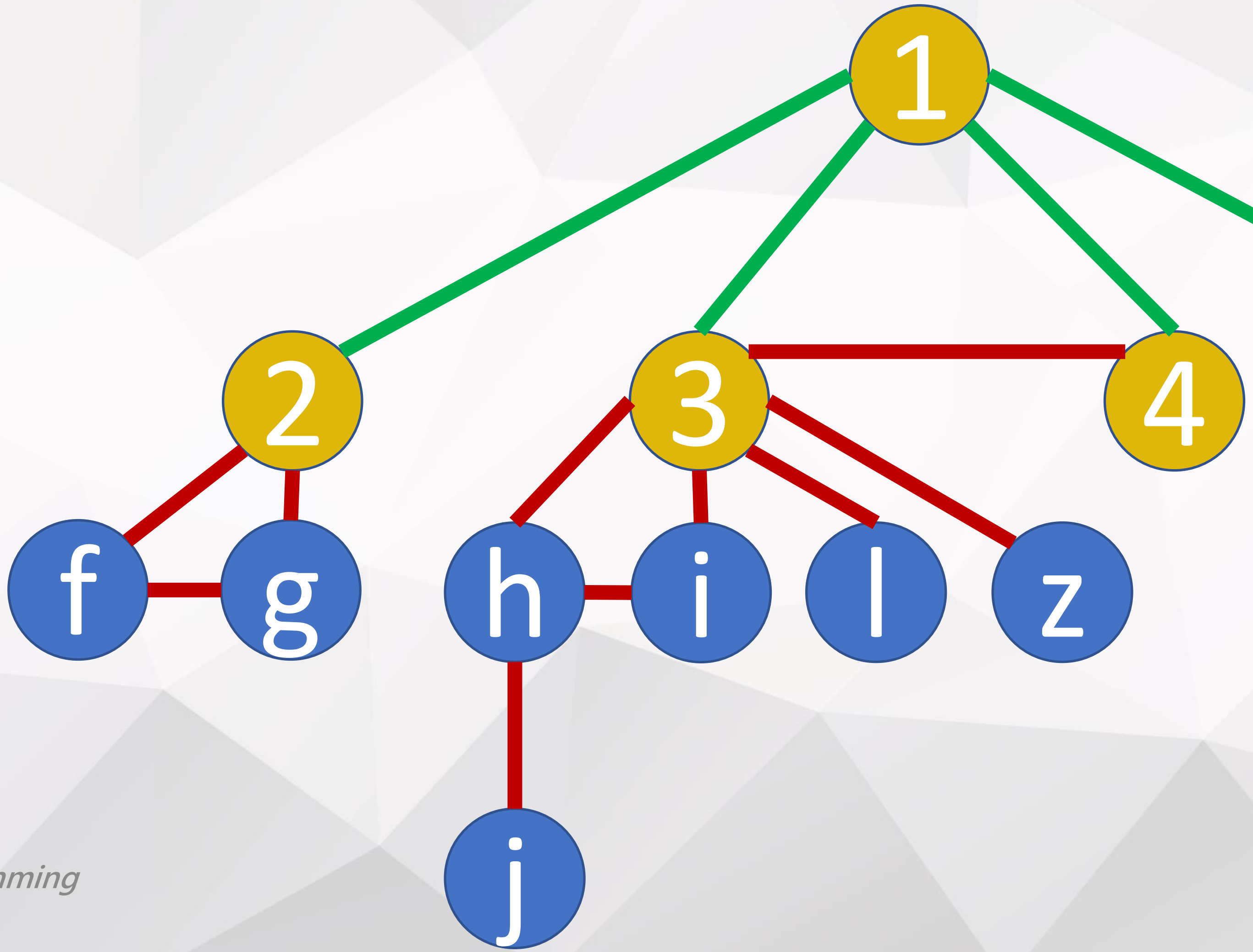
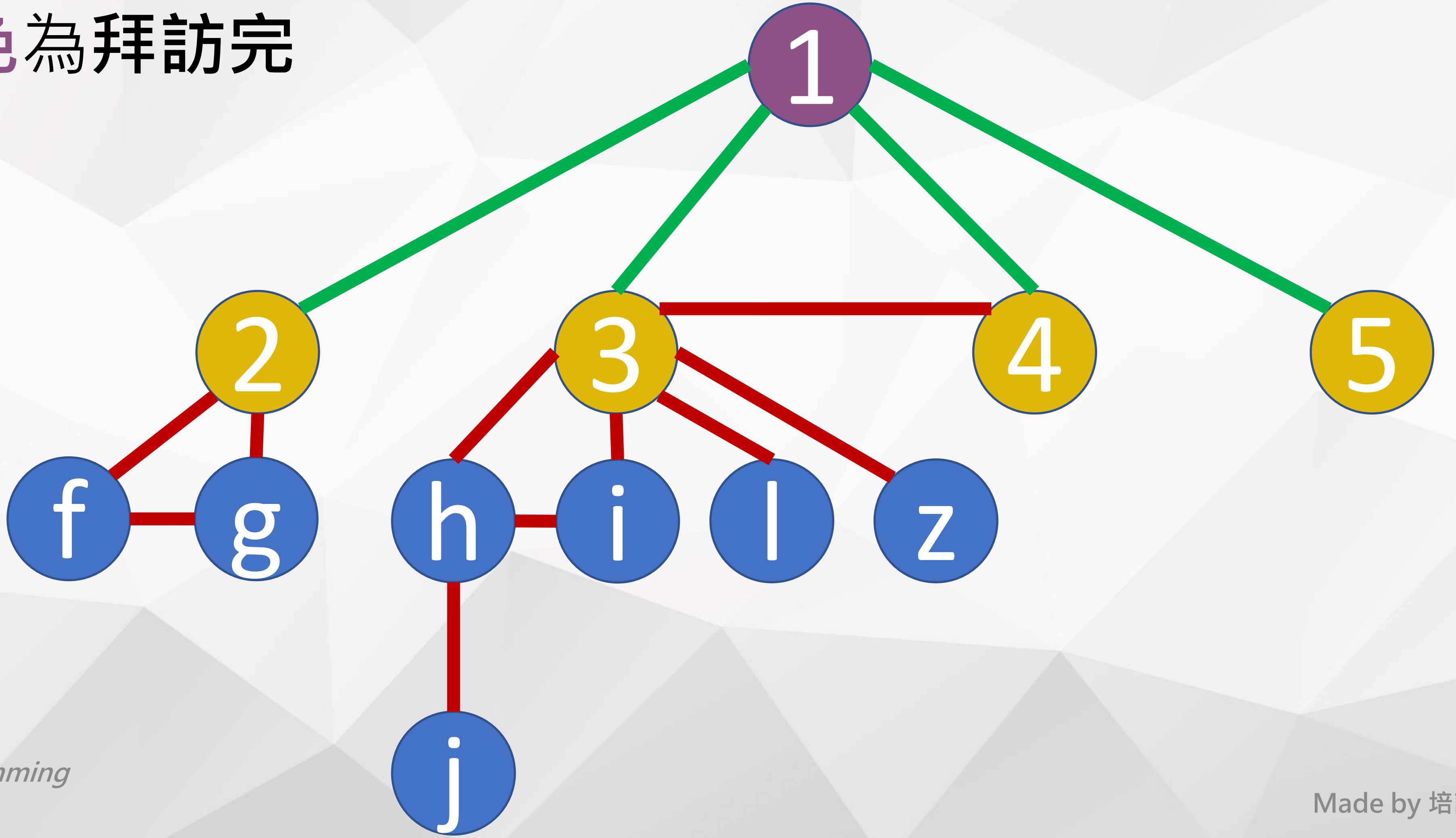


根拜訪完

紫色為拜訪完



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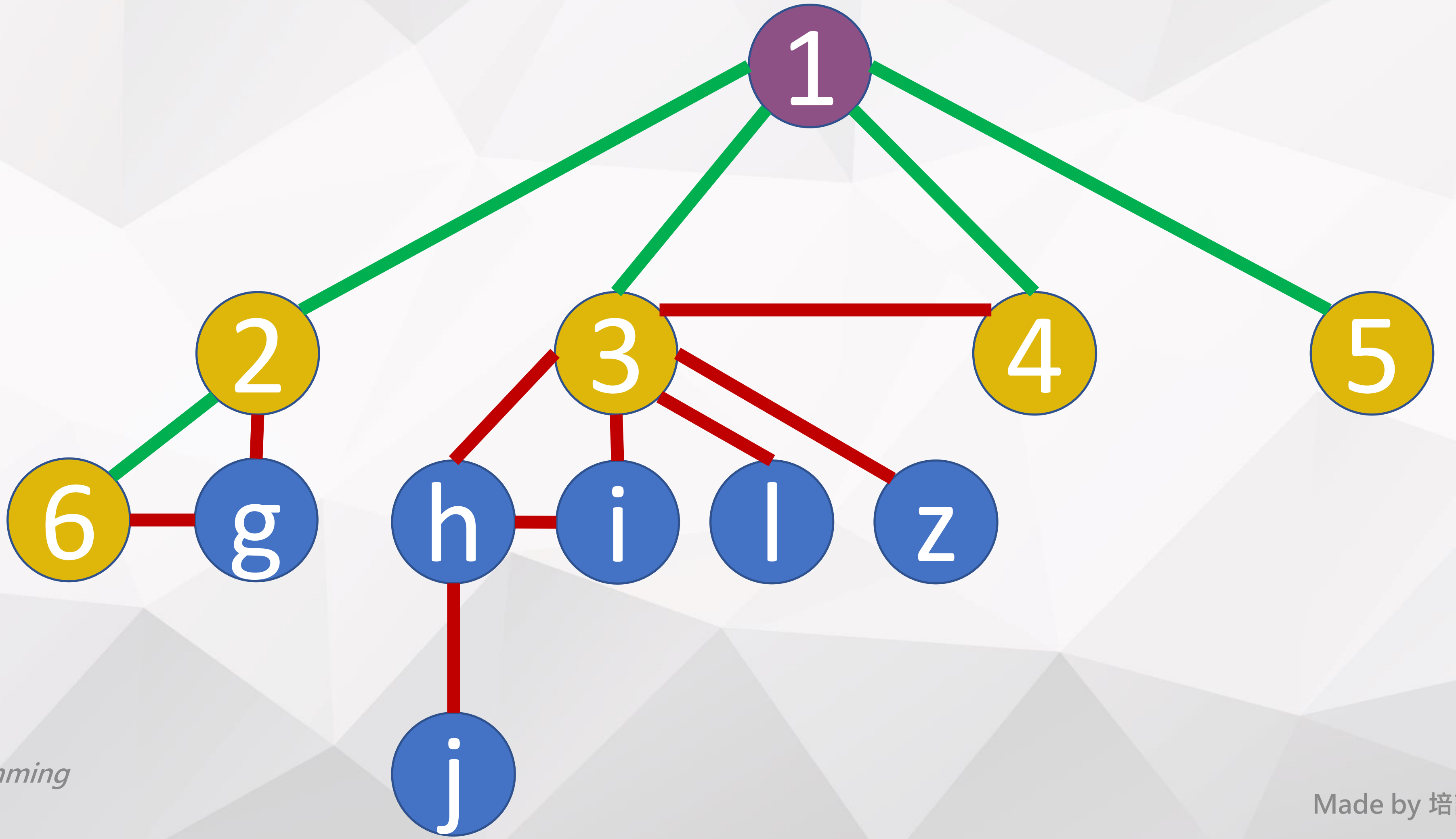

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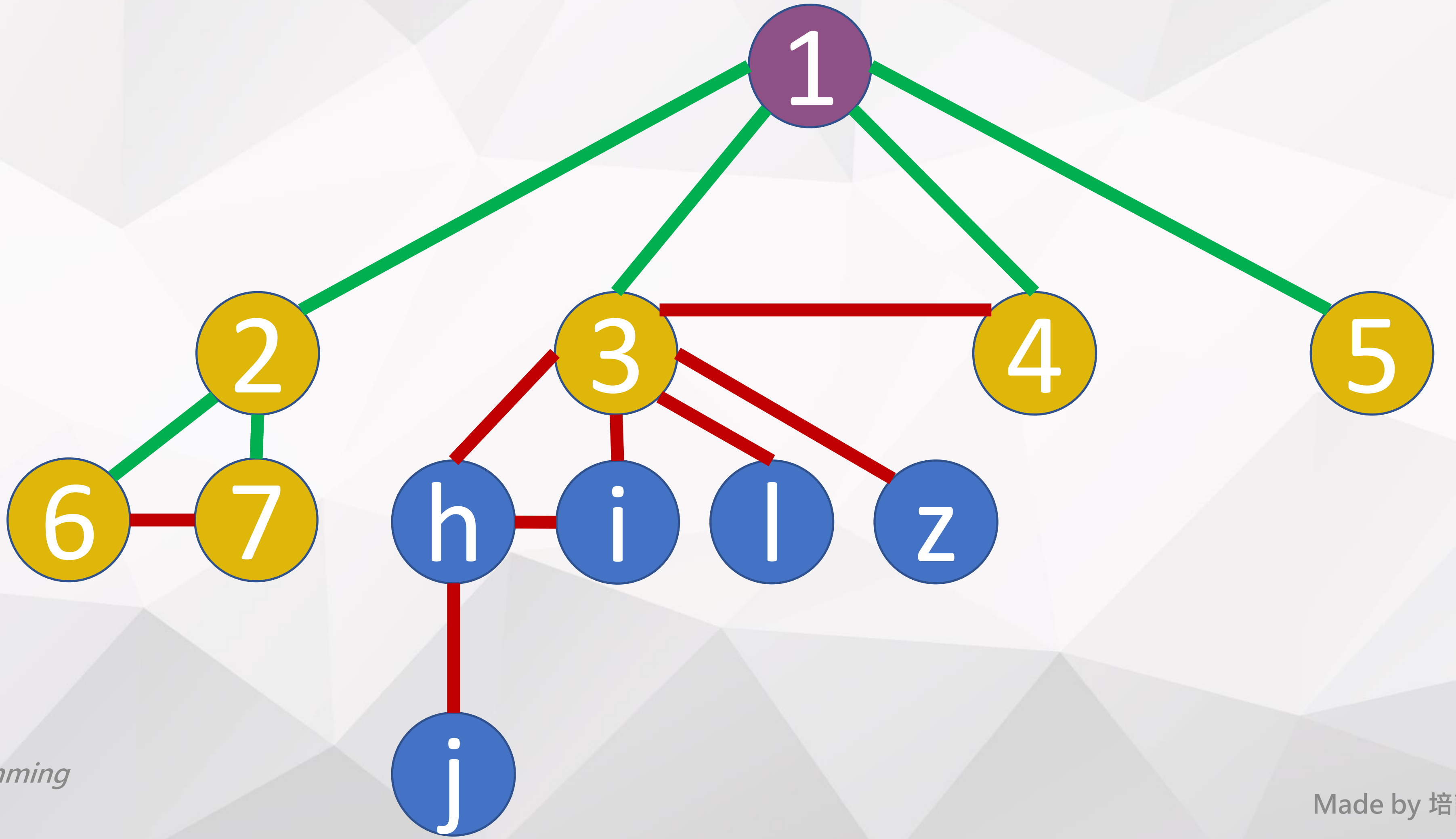

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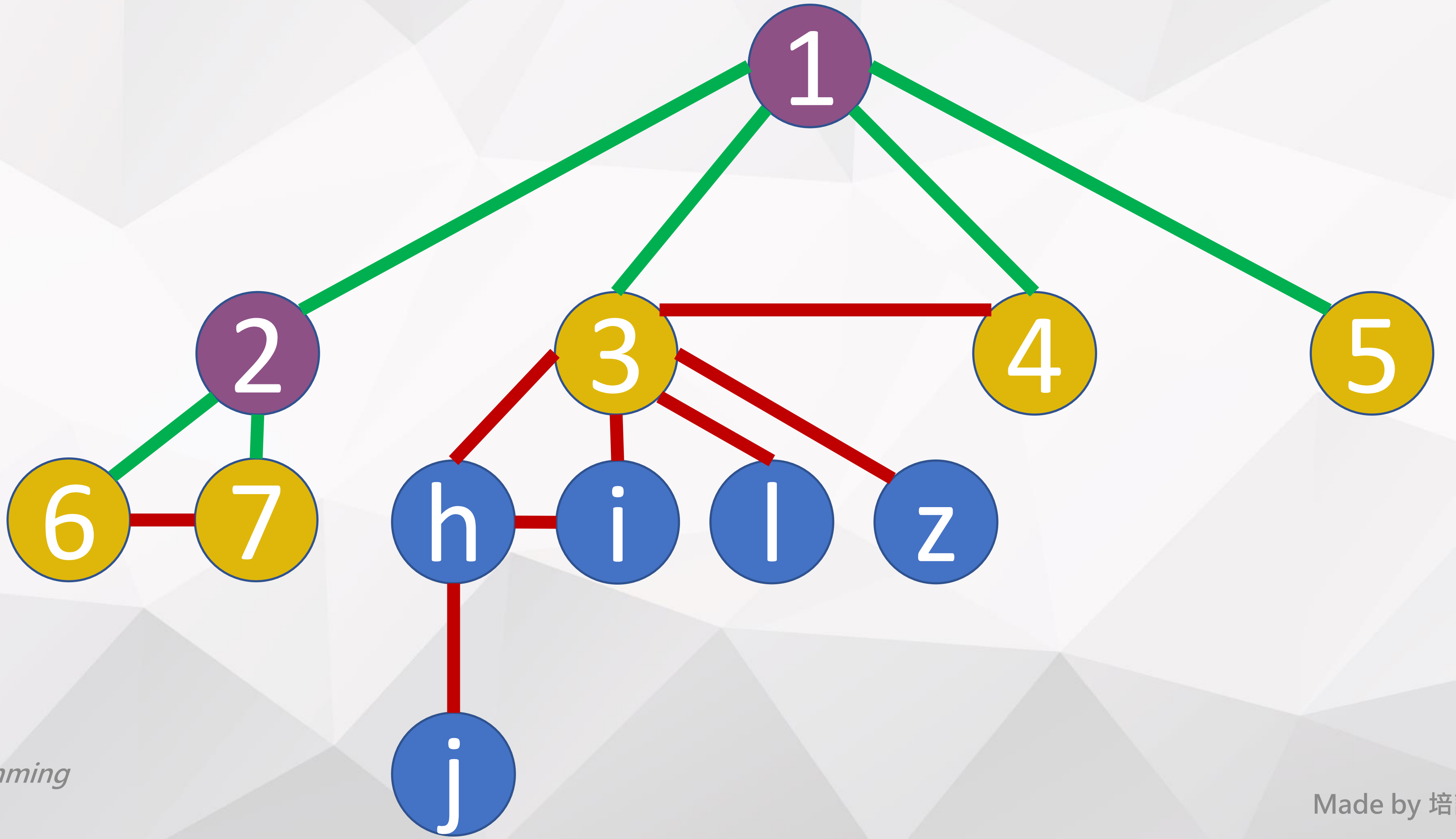

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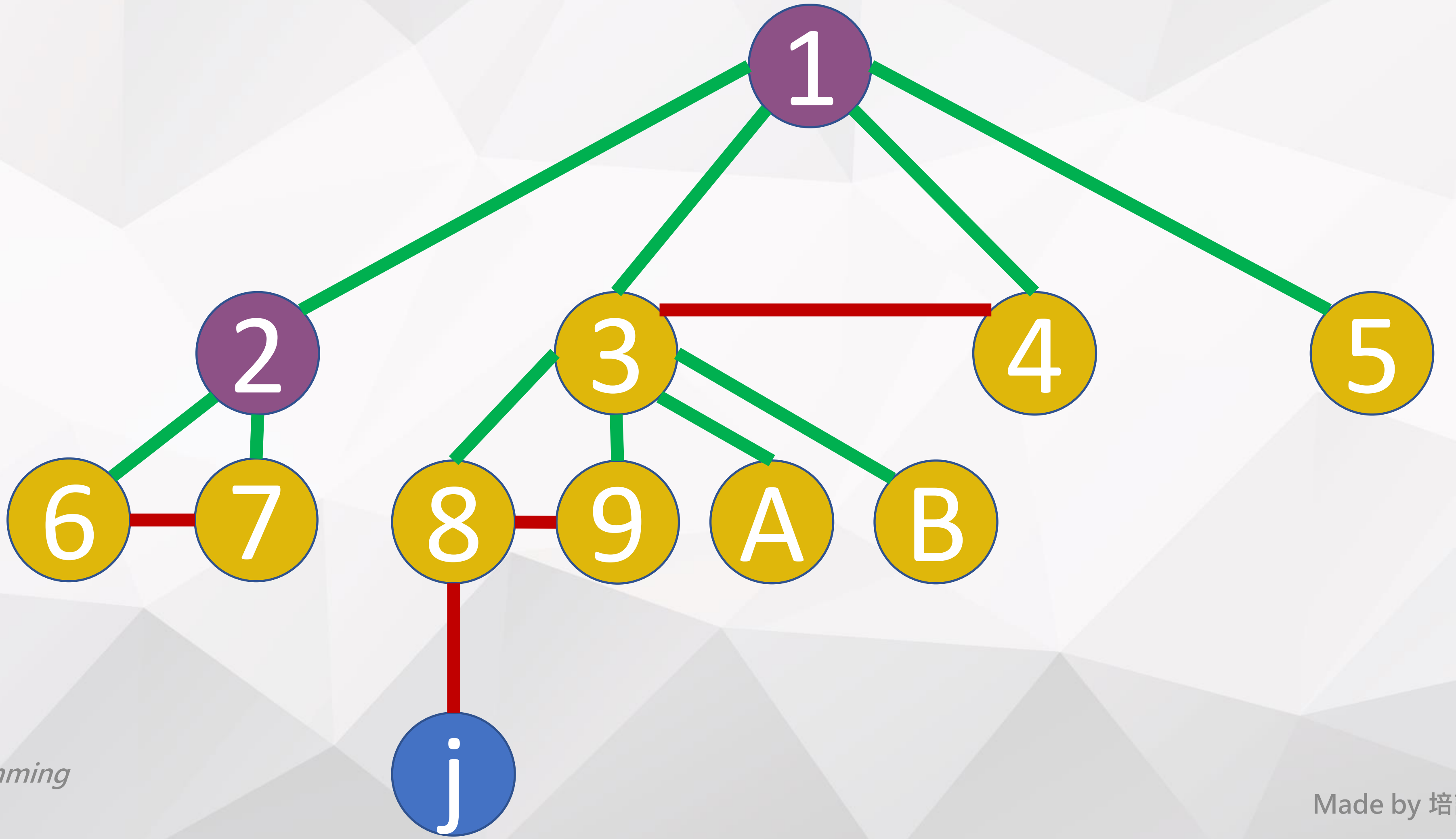

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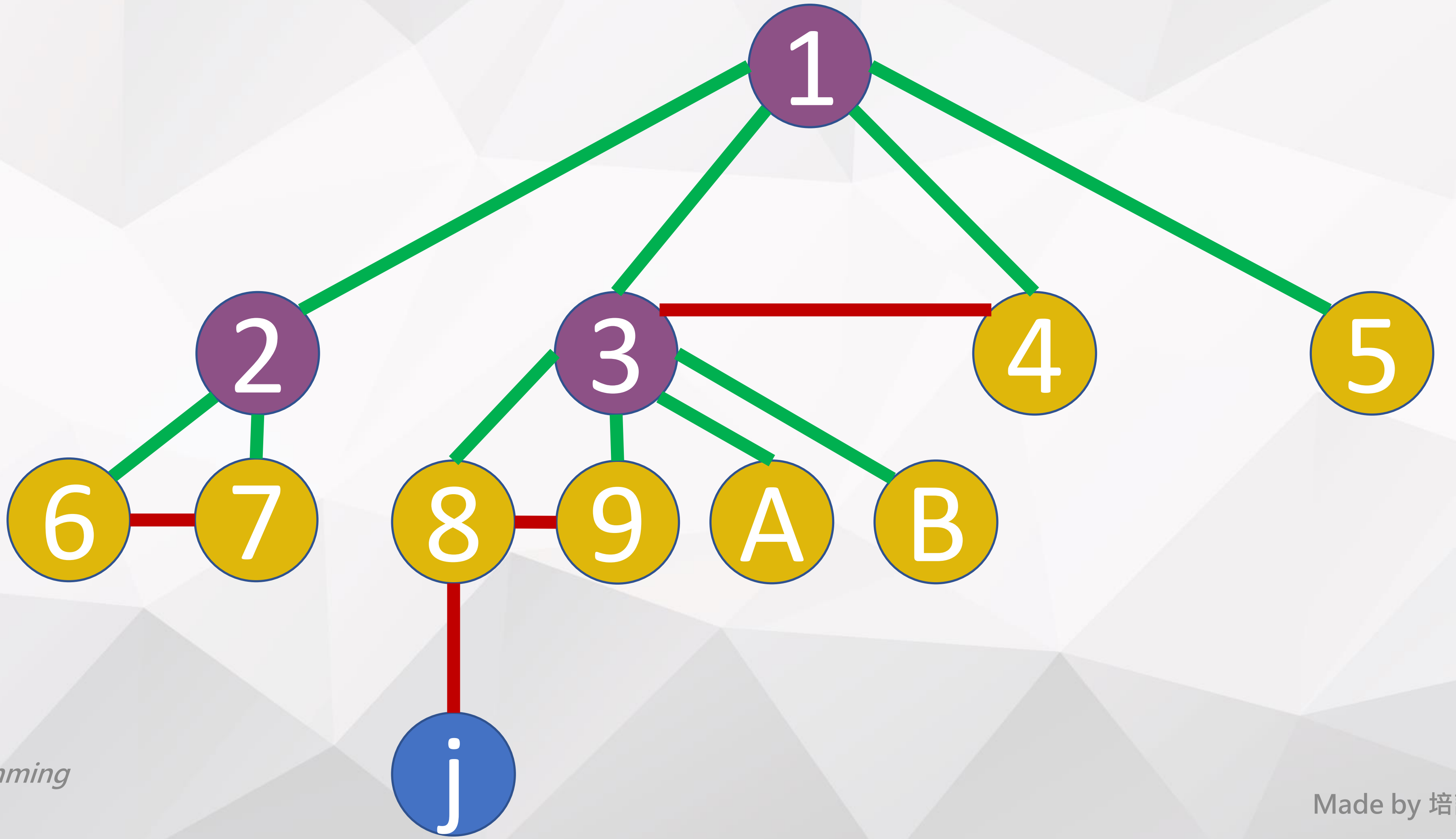

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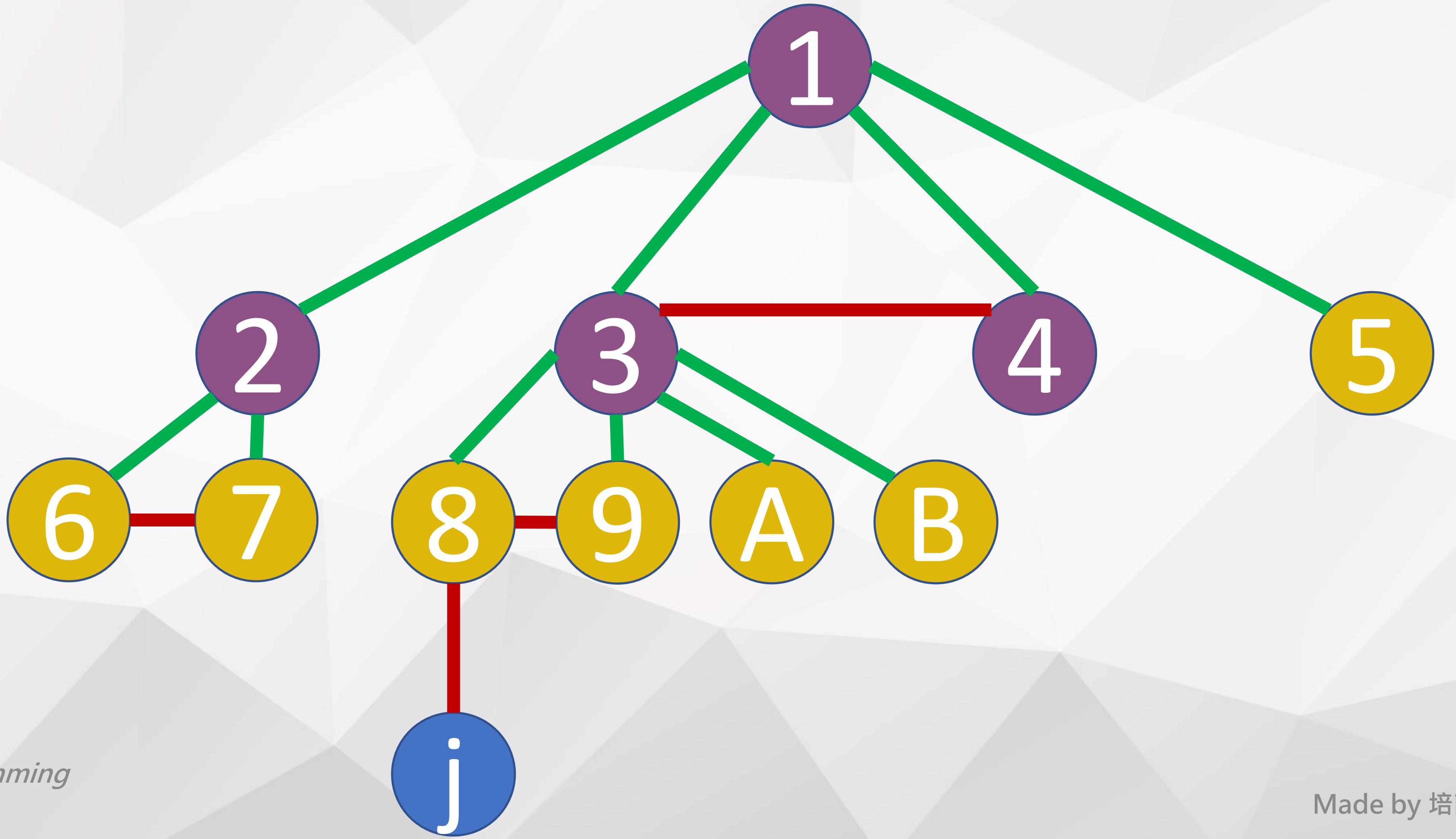

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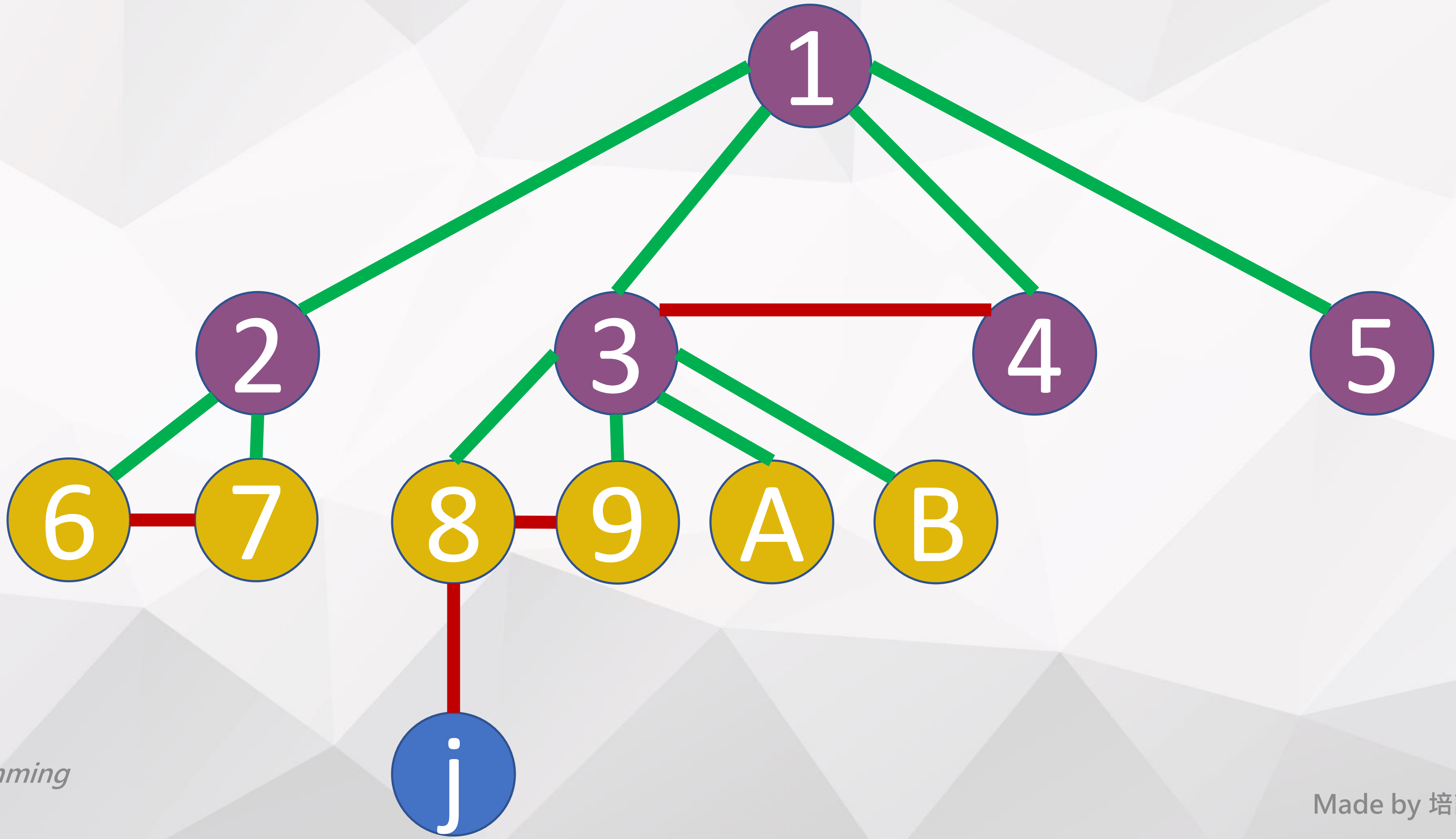

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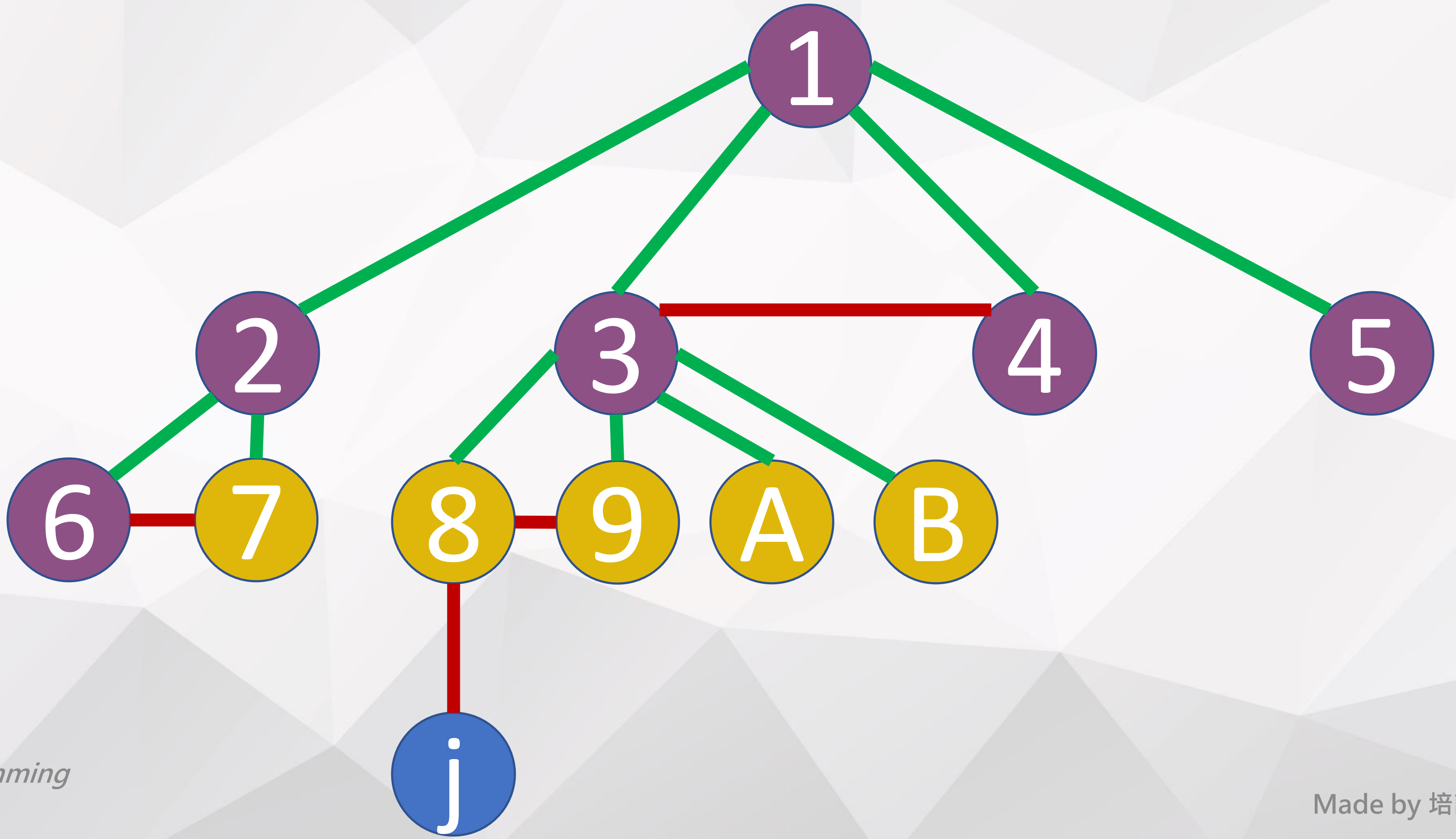

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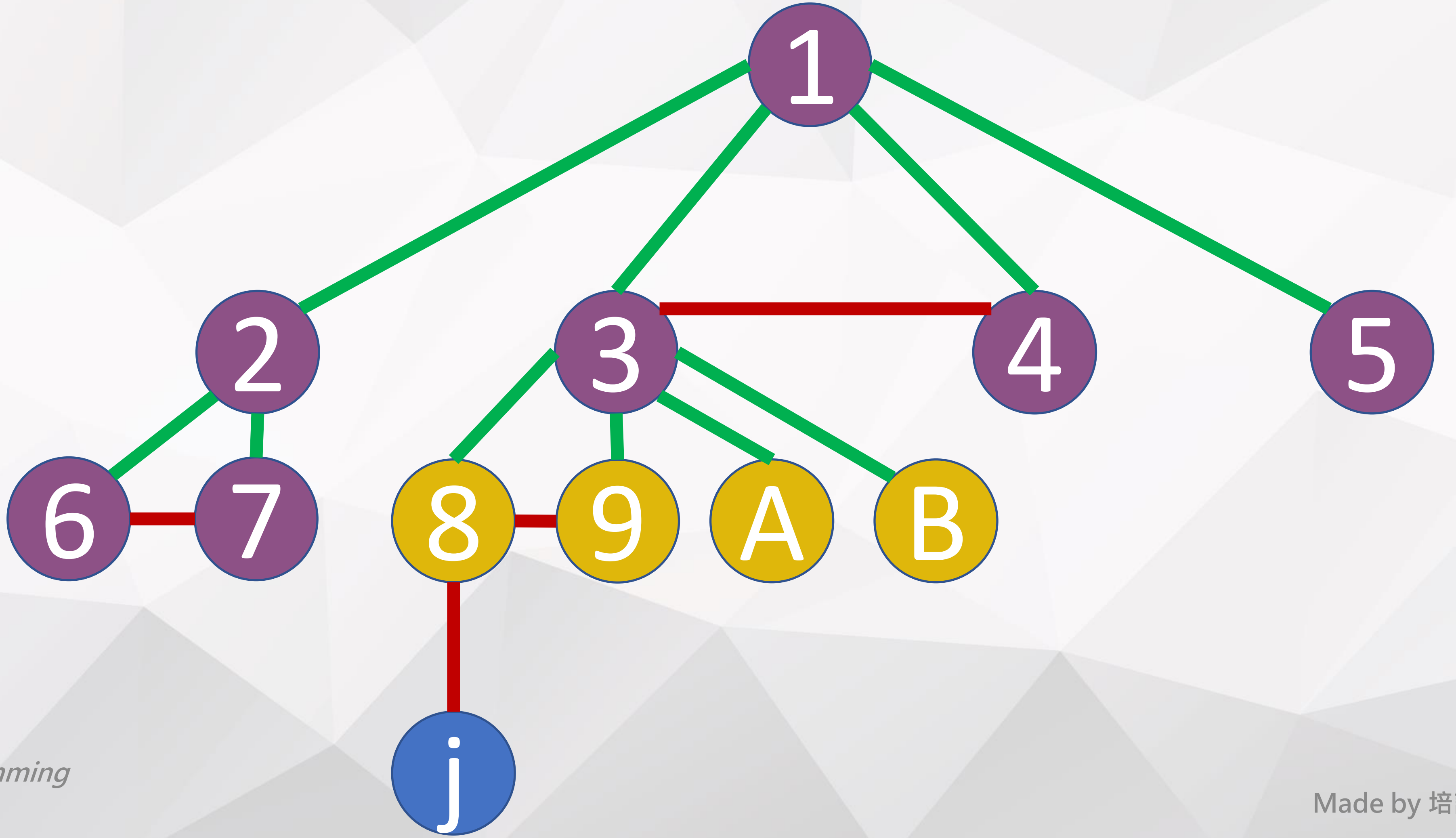

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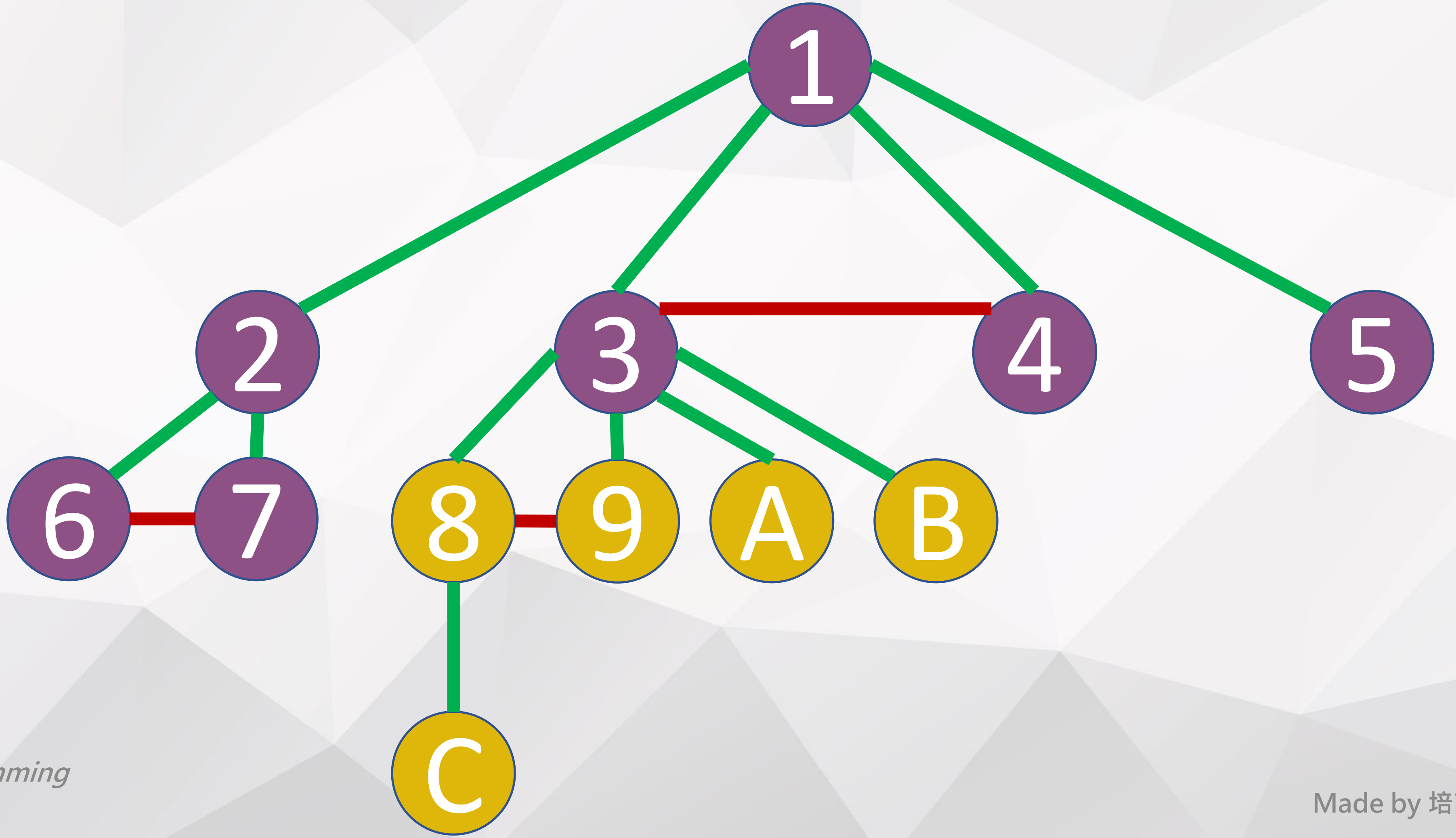

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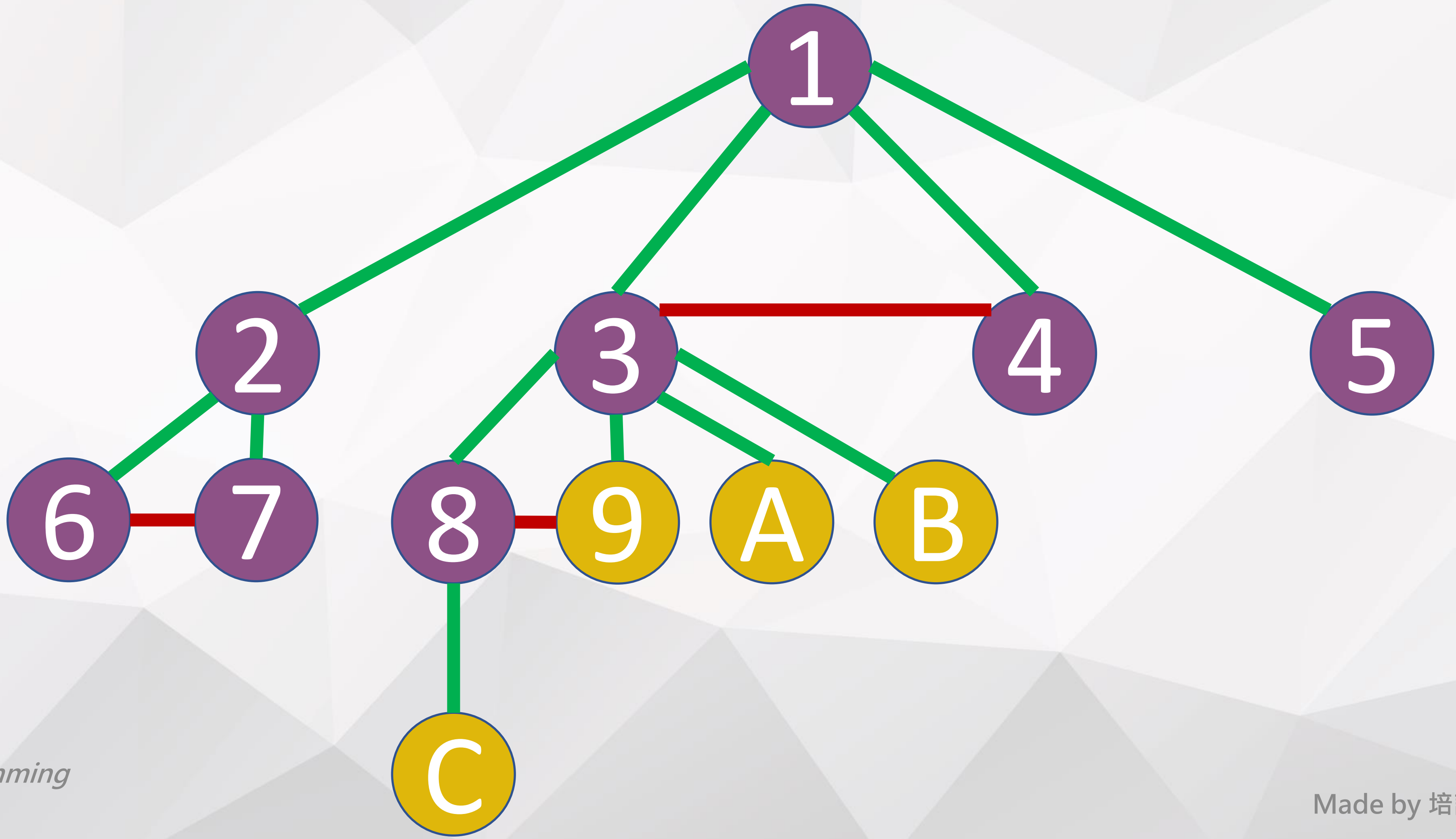

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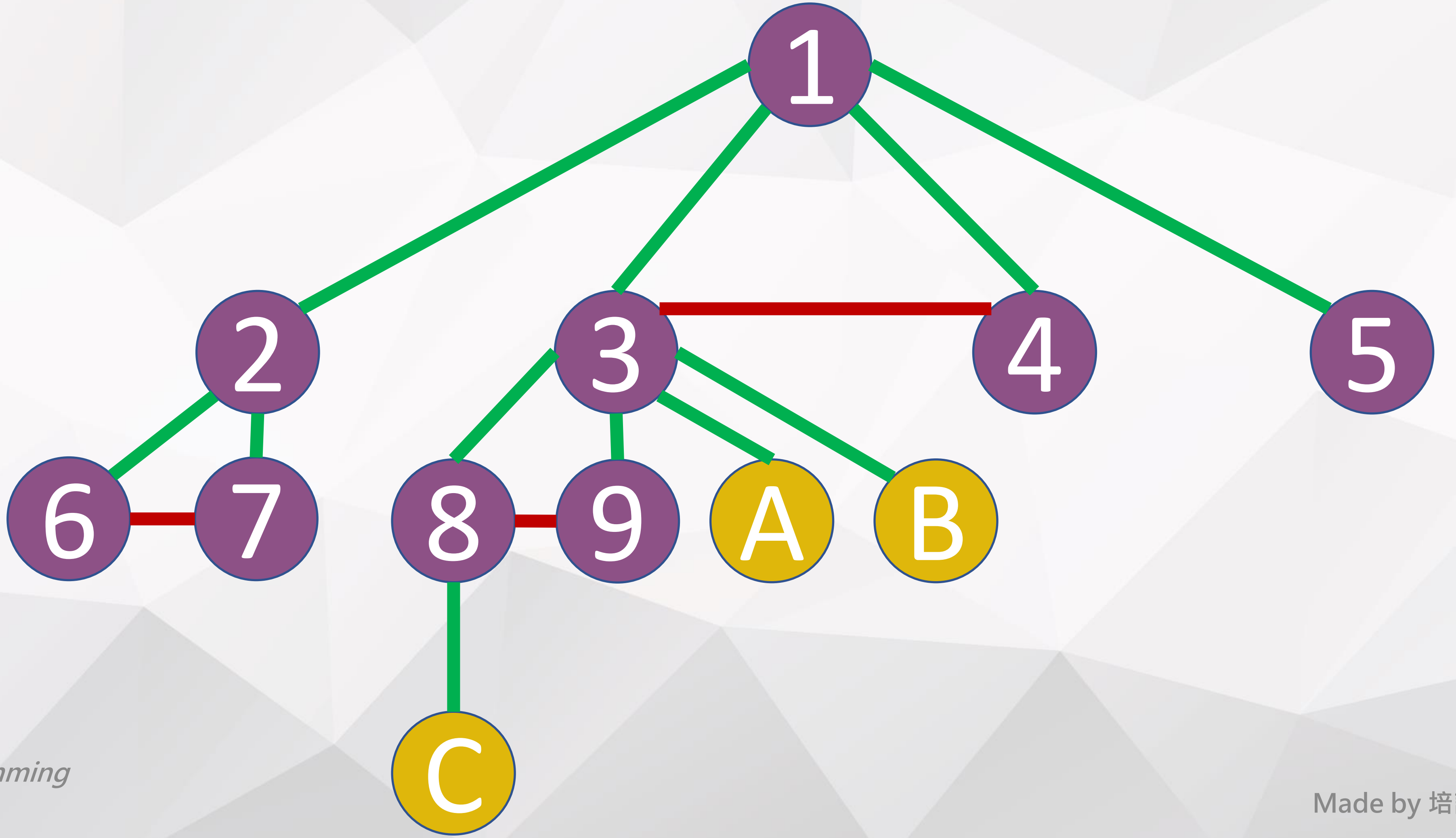

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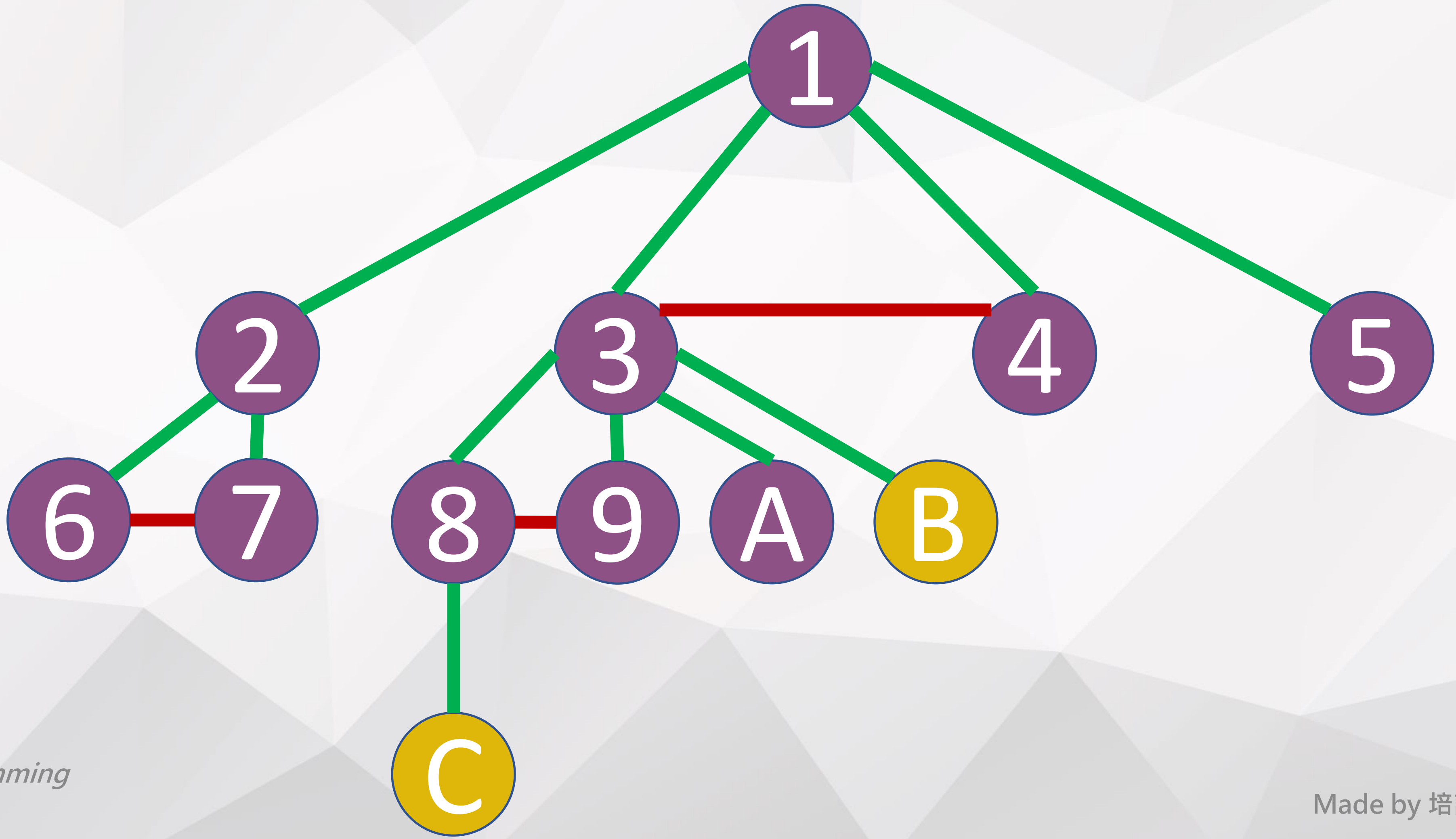

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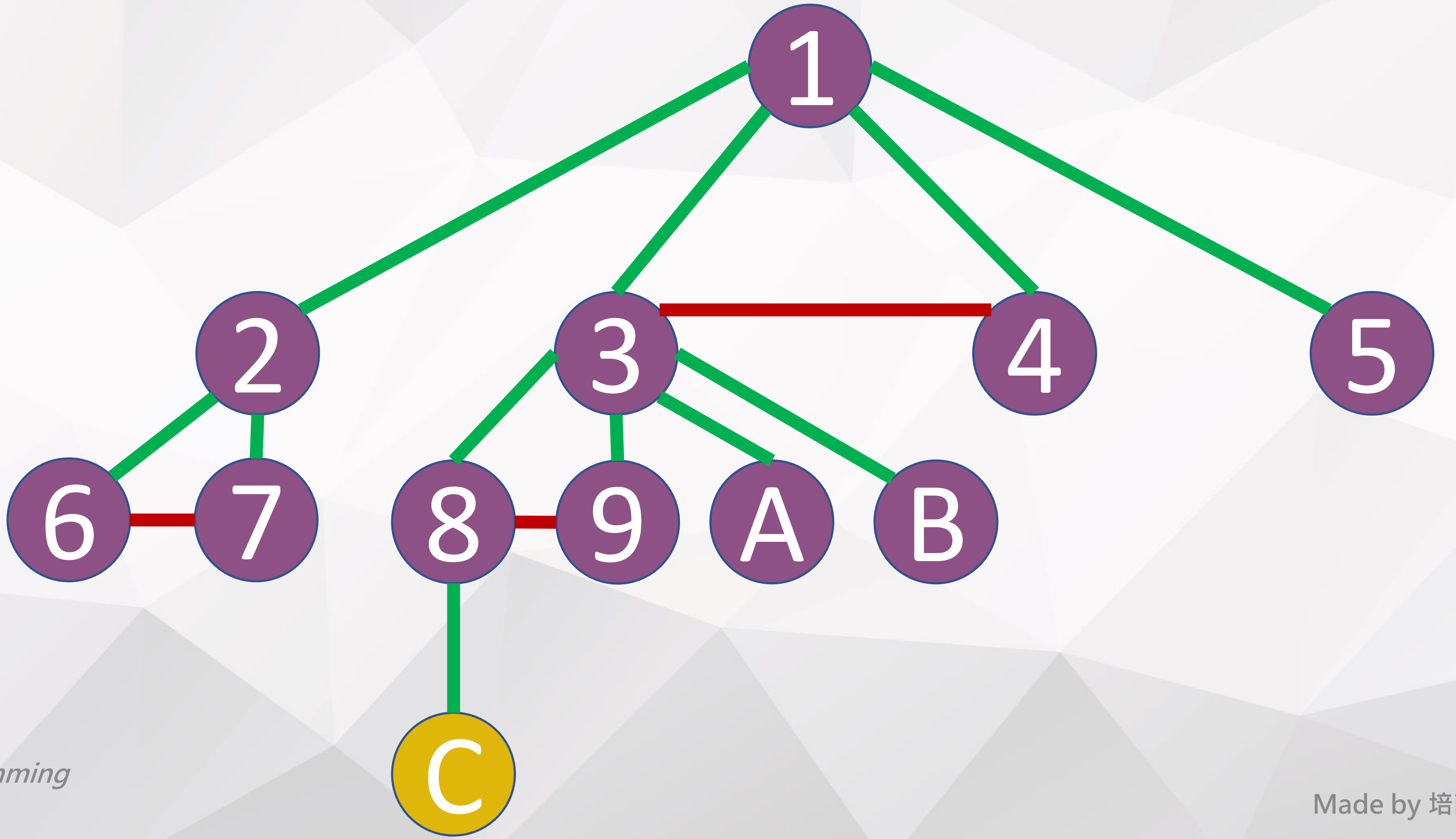

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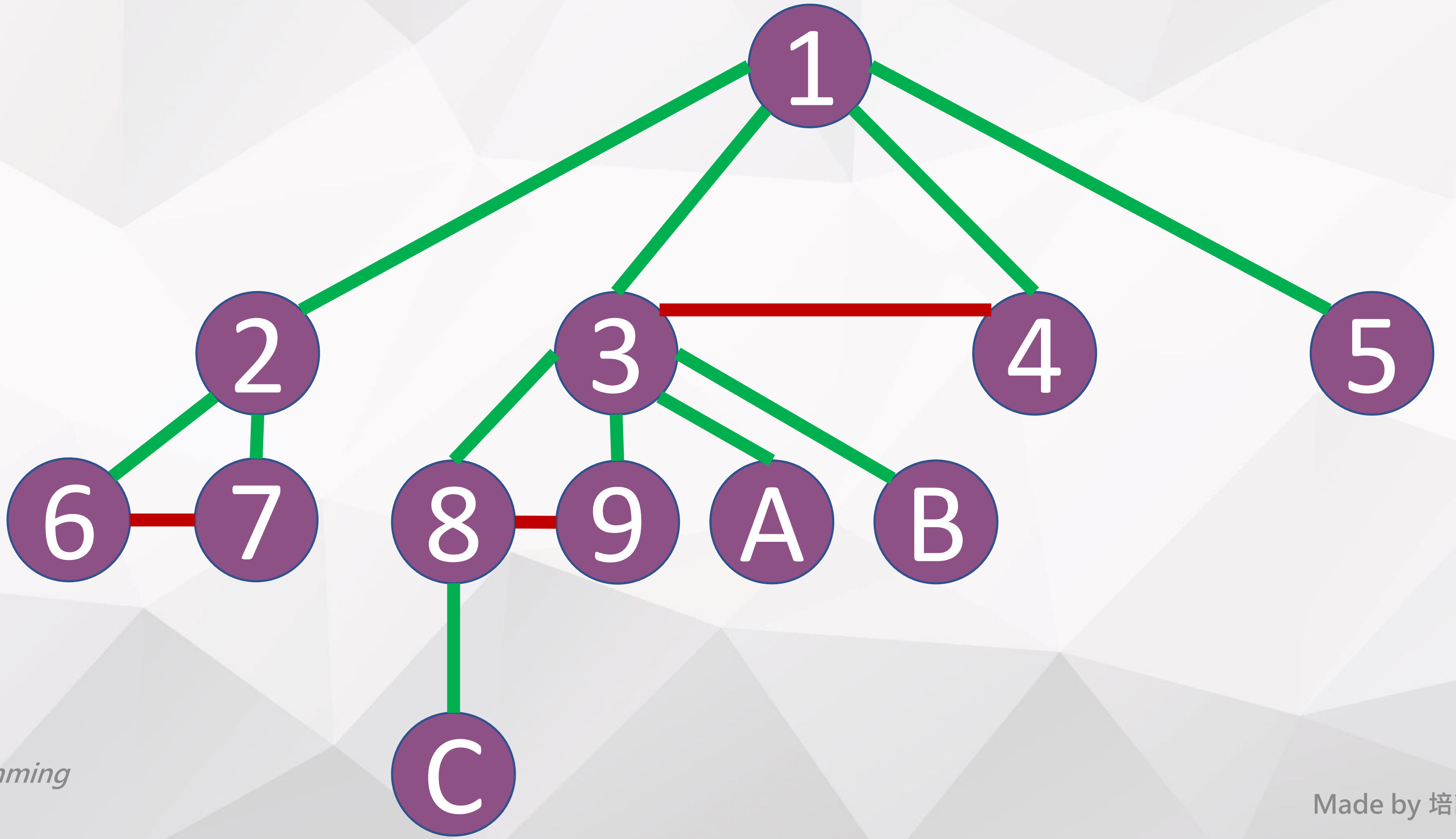
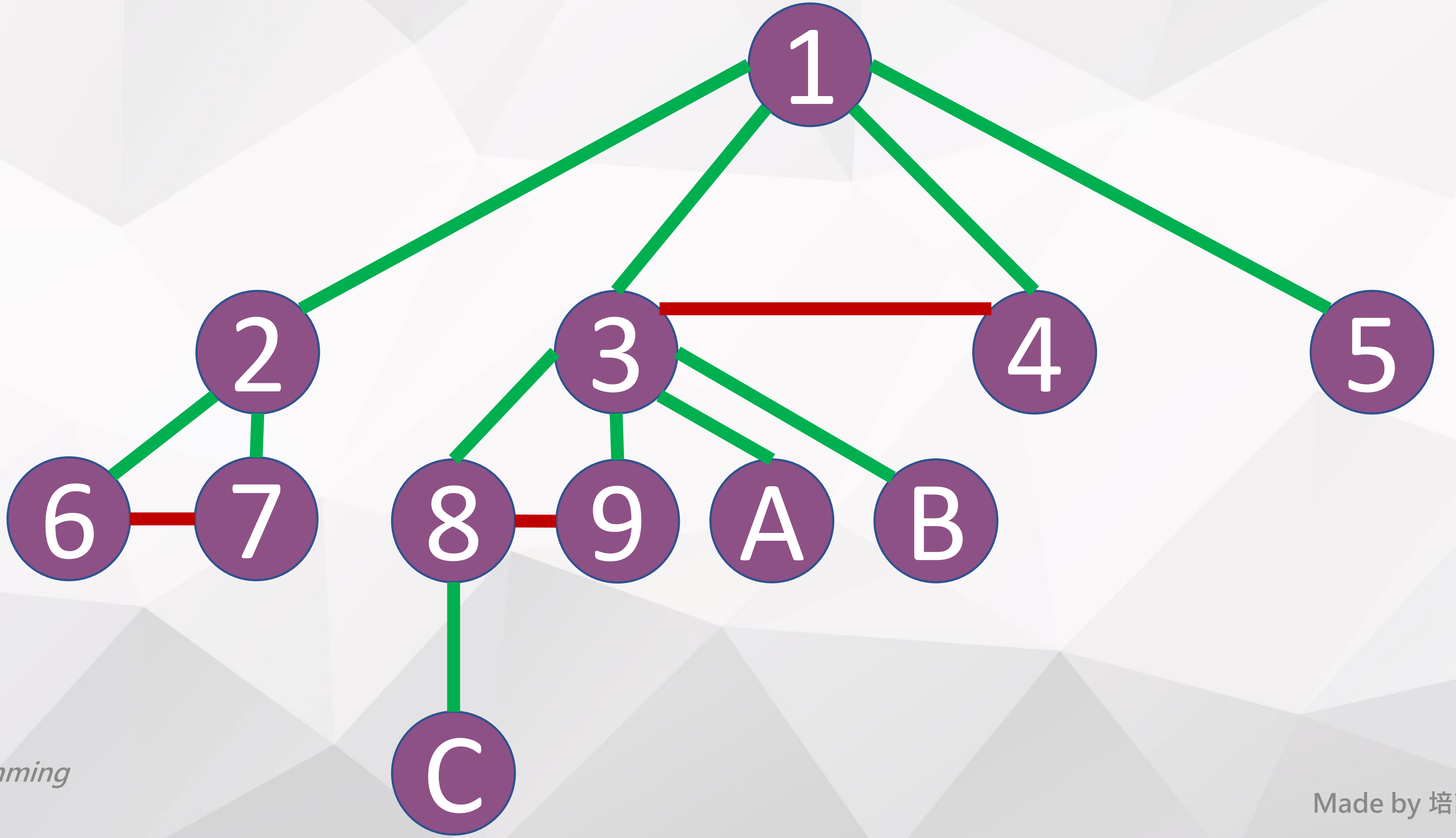
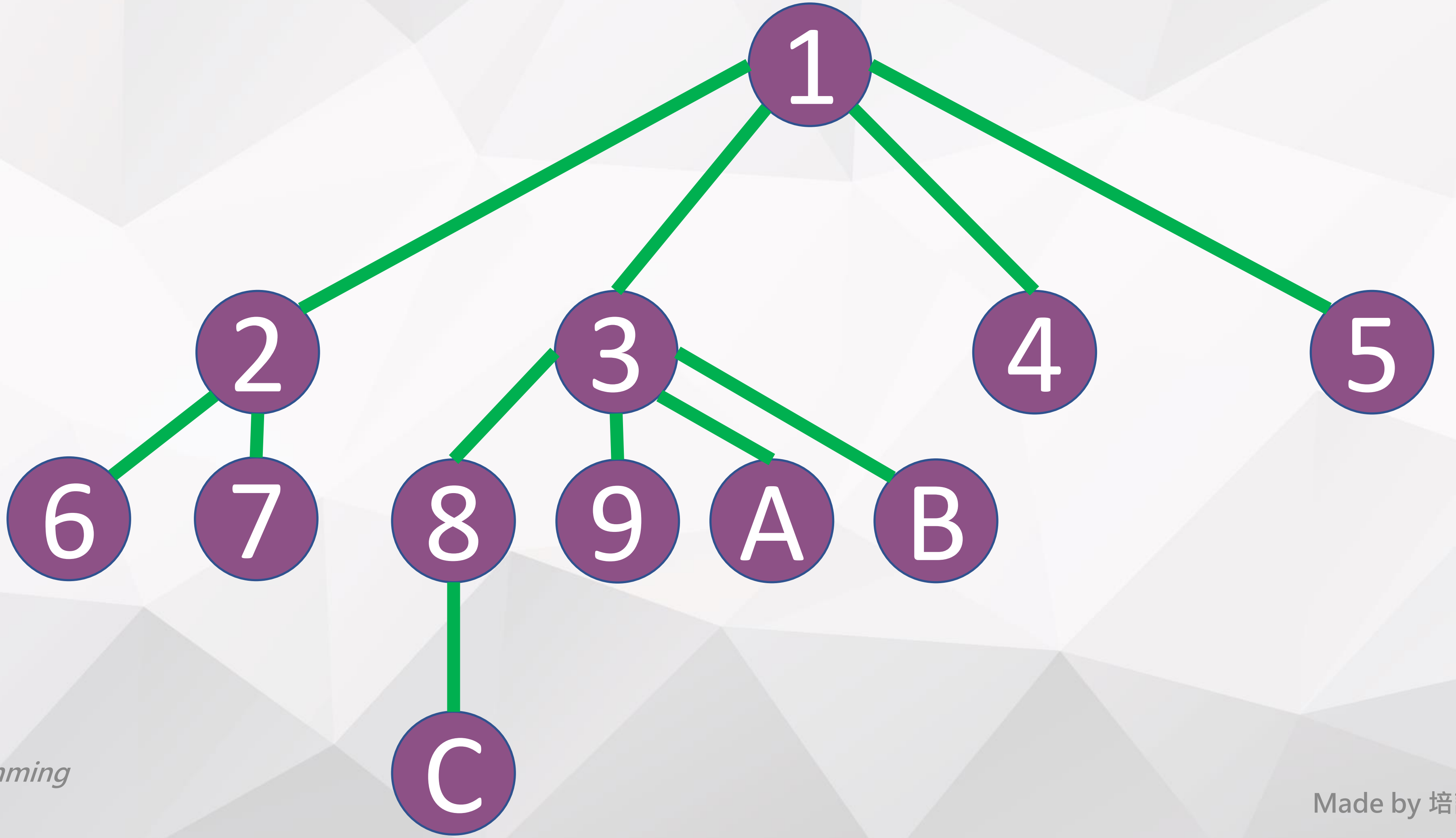


BFS 樹



BFS 樹



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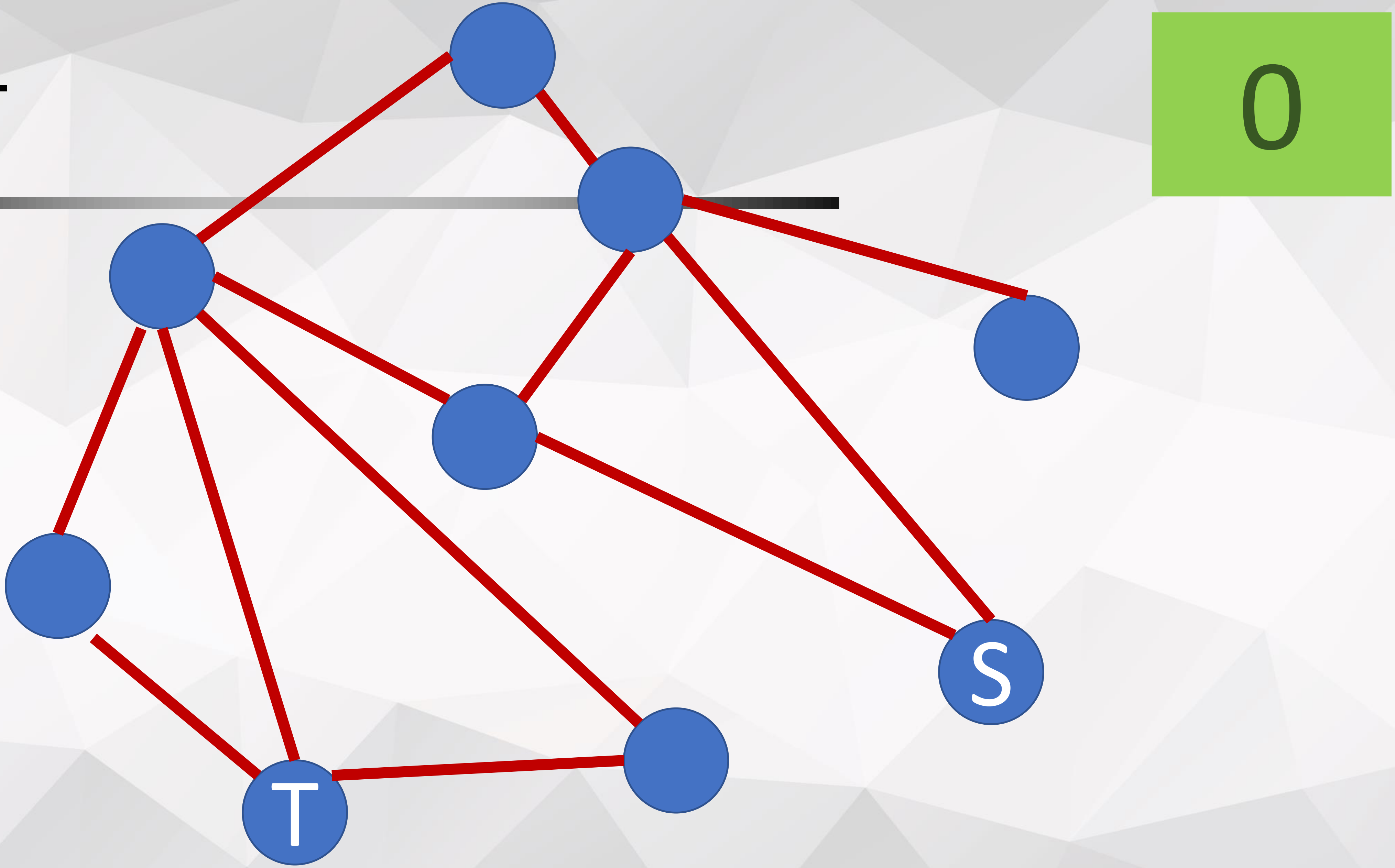

BFS 完

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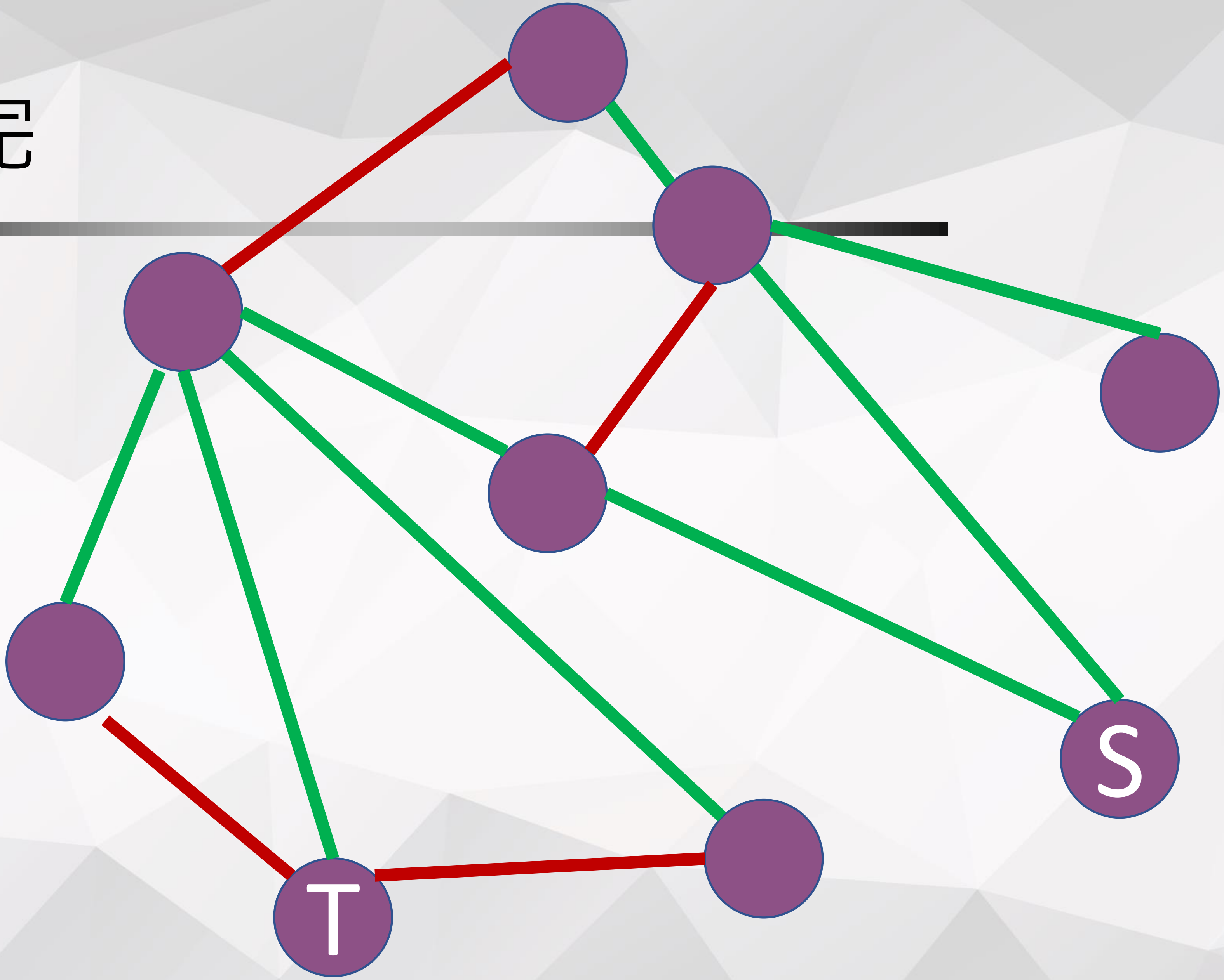

BFS 完

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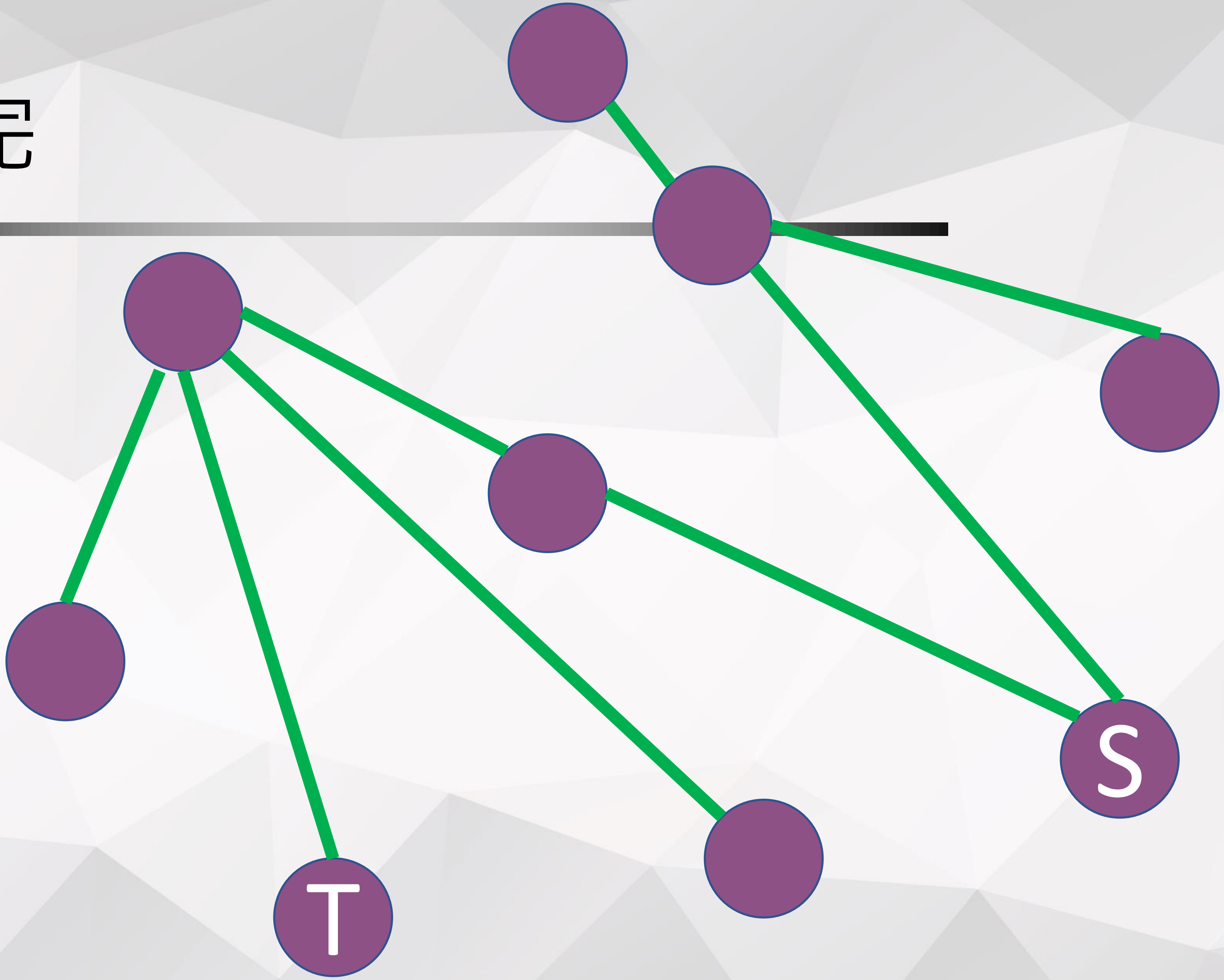
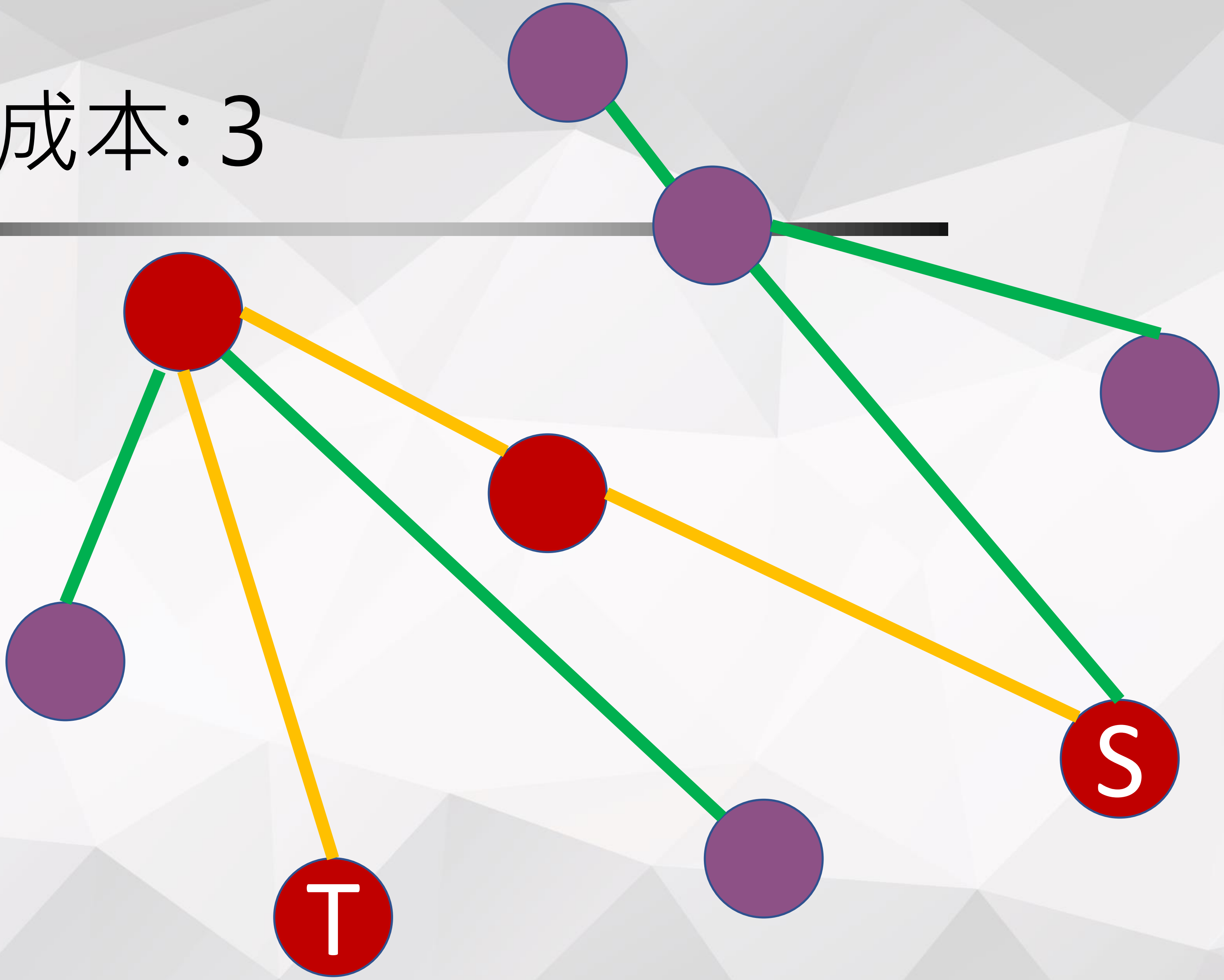
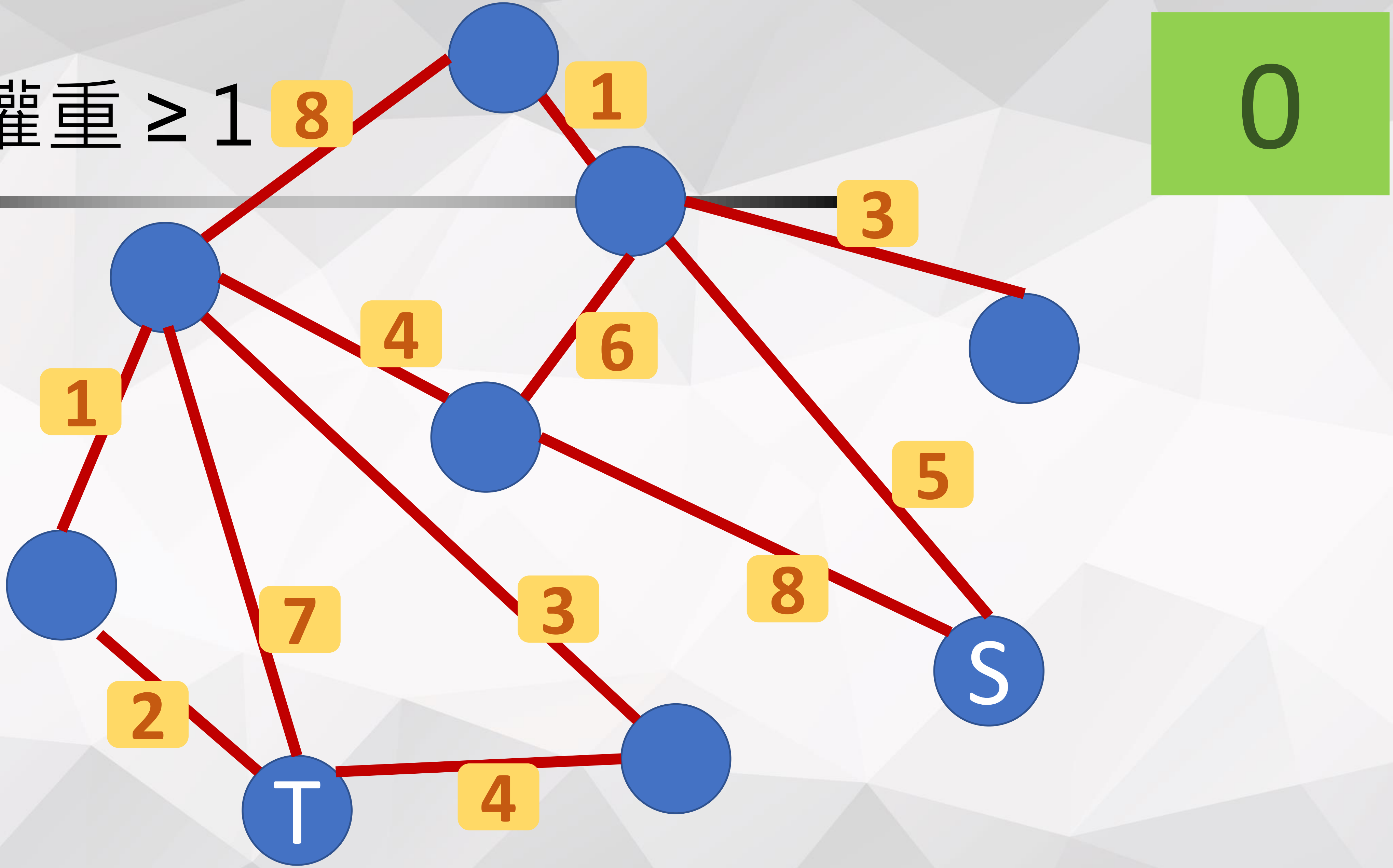


深度/成本: 3

3



若邊權重 ≥ 1



若邊權重 ≥ 1

怎麼辦?

將權重切段

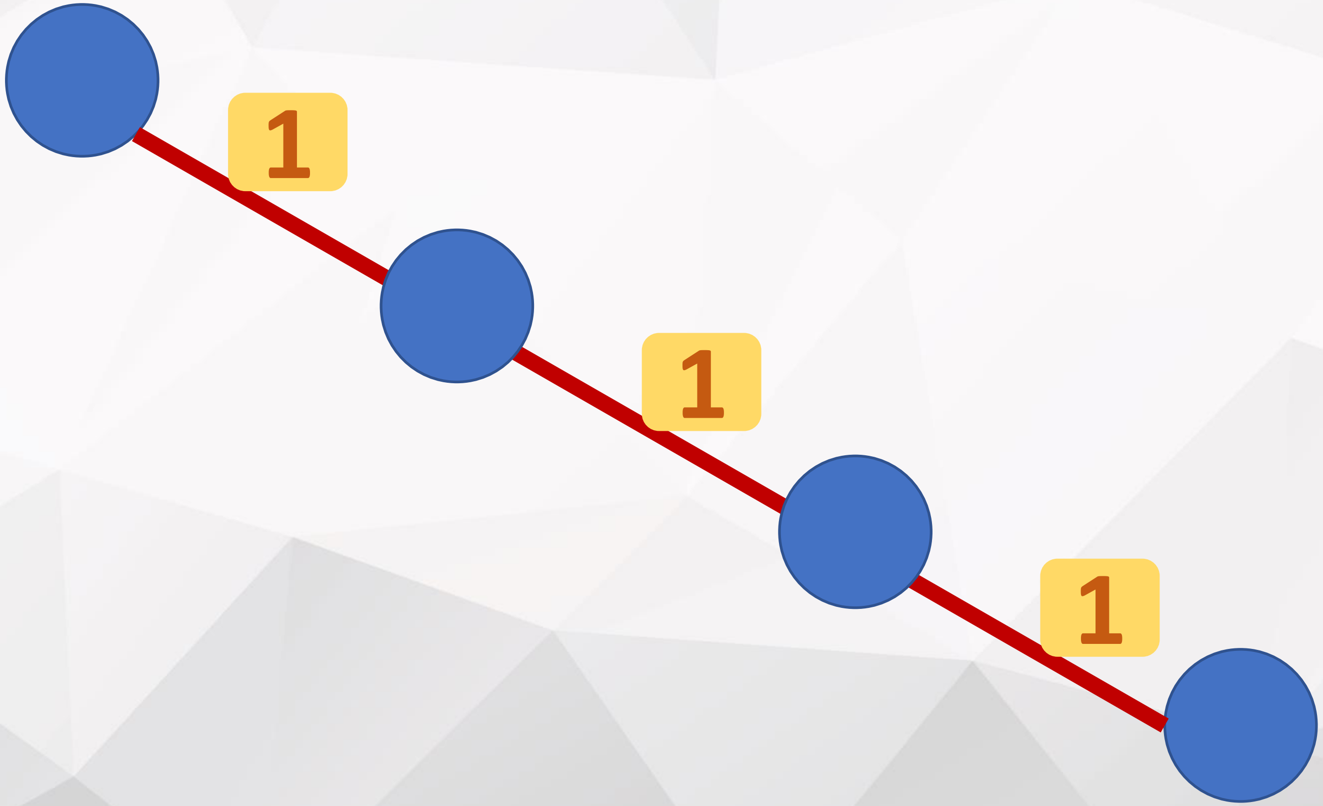
例如 x 權重，切成共 x 段的 1 權重邊

將權重切段

例如 3 權重

將權重切段

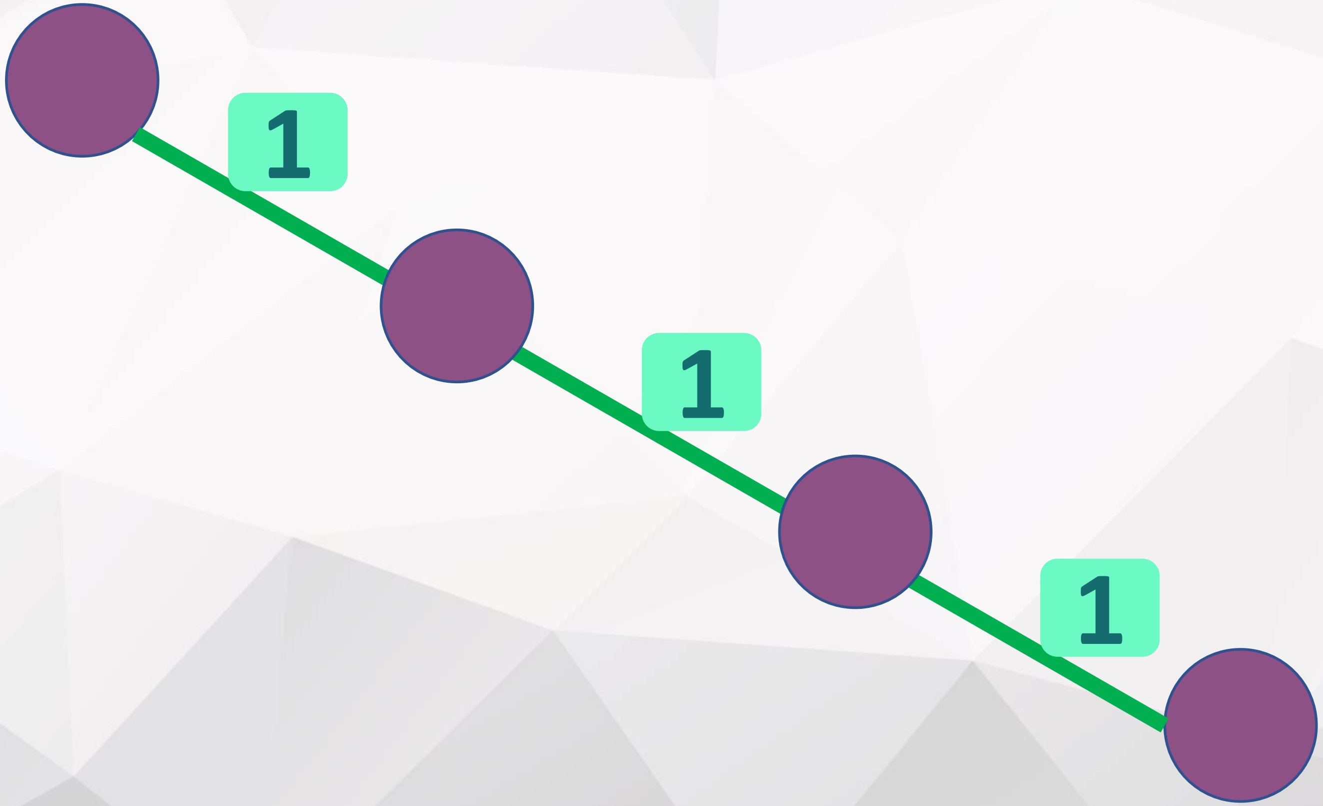
例如 3 權重，切成共 3 段的 1 權重邊



將權重切段

例如 3 權重，切成共 3 段的 1 權重邊

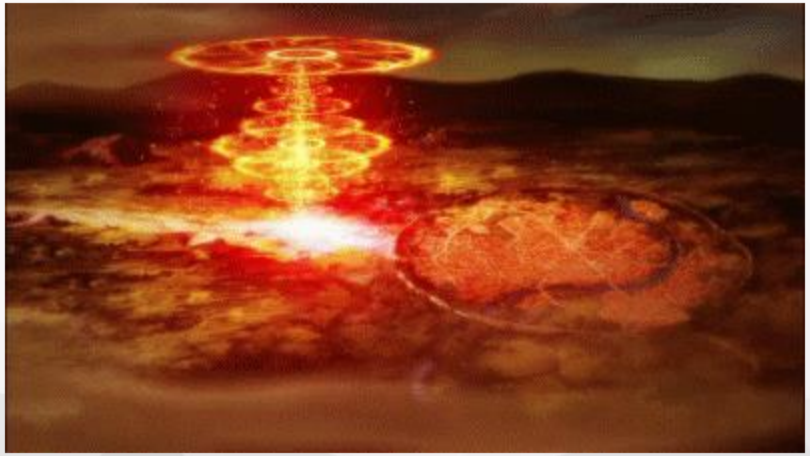
就能 BFS 了!



將權重切段

例如 3 權重，切成共 3 段的 1 權重邊就能 BFS 了!

複雜度肯定會爆炸!



Questions?

單源最短路徑

- Relaxation
- Dijkstra's algorithm
- Bellman-Ford's algorithm

單源最短路徑

- Relaxation
- Dijkstra's algorithm
- Bellman-Ford's algorithm

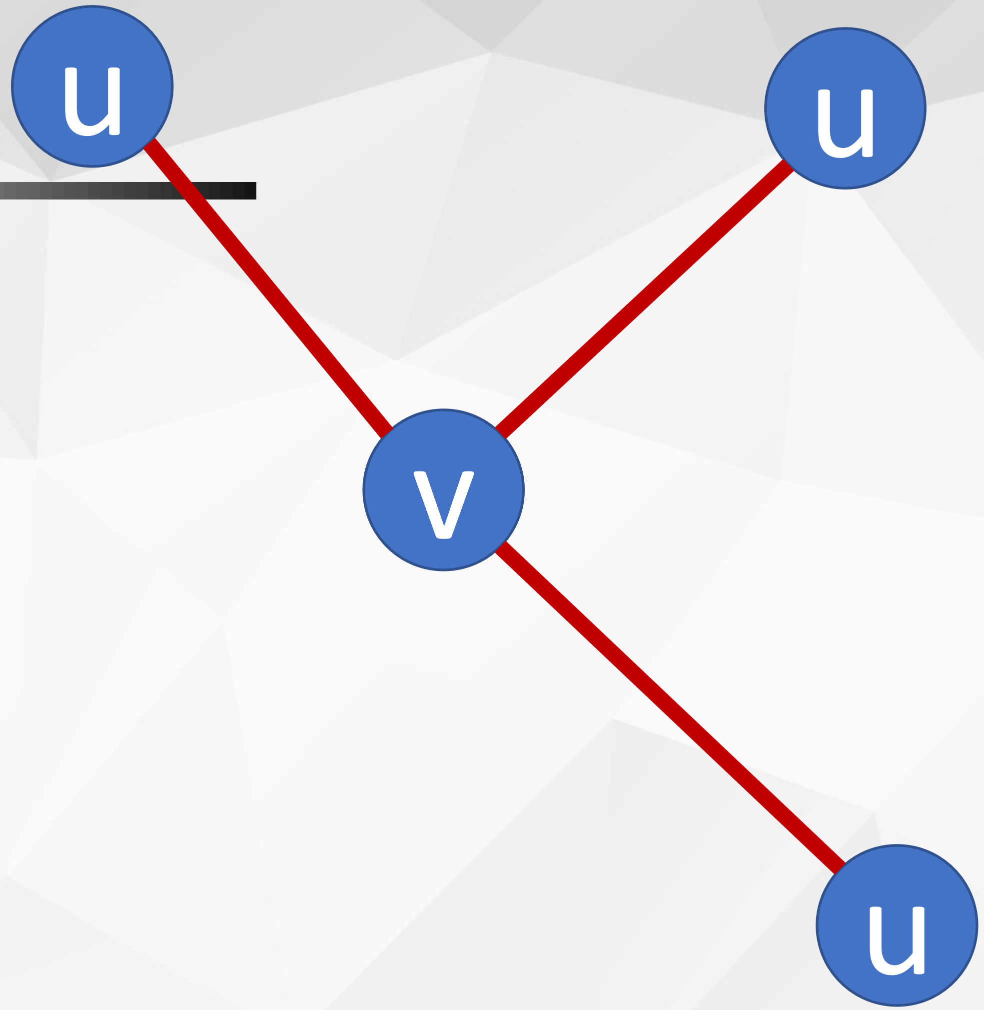
Relaxation

想像自己是圖中的某點 v

Relaxation

想像自己是圖中的某點 v

身旁有一些鄰點 u

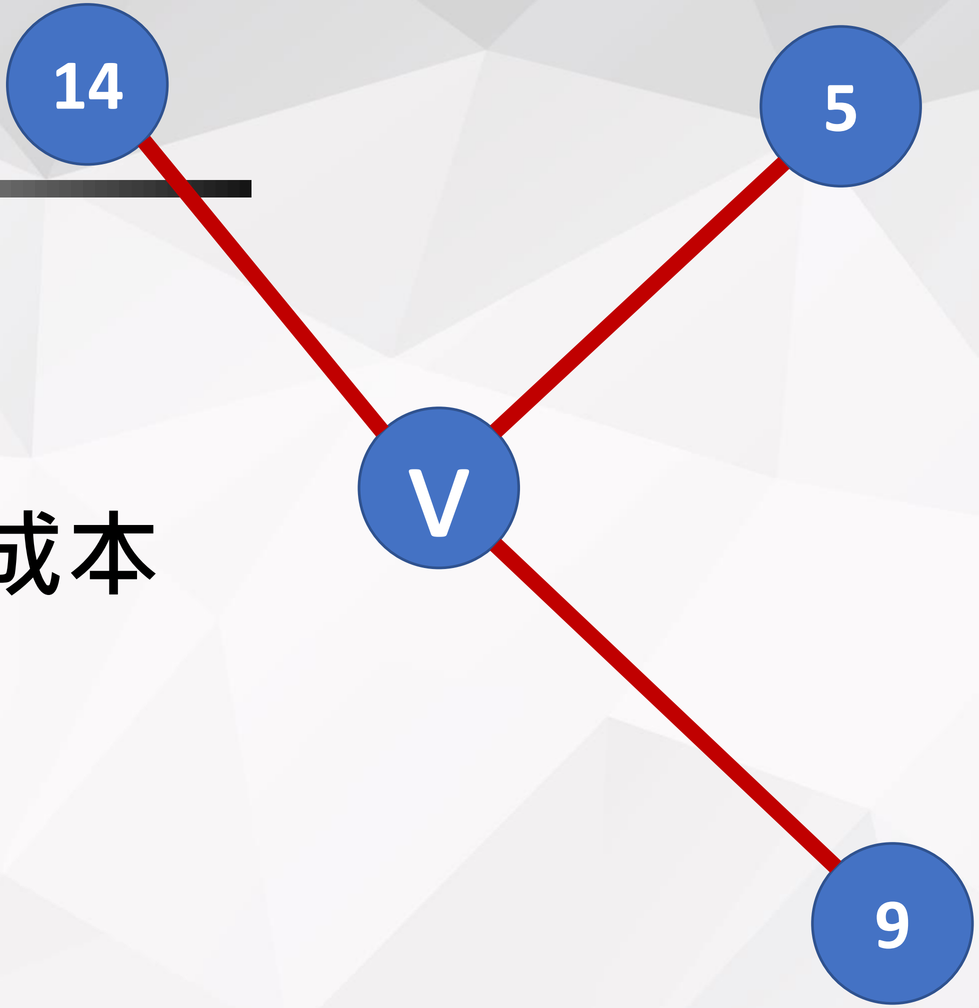


Relaxation

想像自己是圖中的某點 v

身旁有一些鄰點 u

u 知道源點到他們那裡的**最小成本**



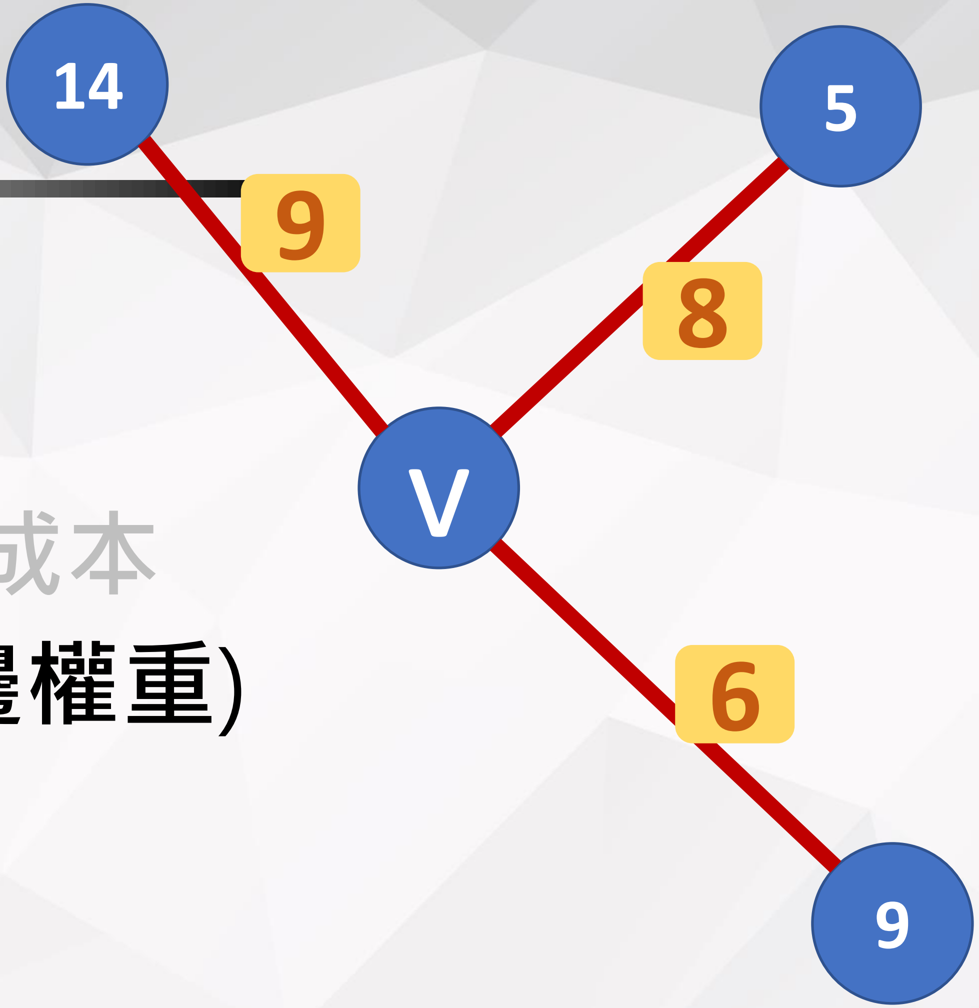
Relaxation

想像自己是圖中的某點 v

身旁有一些鄰點 u

u 知道源點到他們那裡的**最小成本**

v 知道 u 到自己那裡的成本 (**邊權重**)



Relaxation

想像自己是圖中的某點 v

身旁有一些鄰點 u

u 知道源點到他們那裡的**最小成本**

v 知道 u 到自己那裡的成本 (**邊權重**)

v 能計算各點到 v 的成本

Relaxation

想像自己是圖中的某點 v
身旁有一些鄰點 u
u 知道源點到他們那裡的**最小成本**
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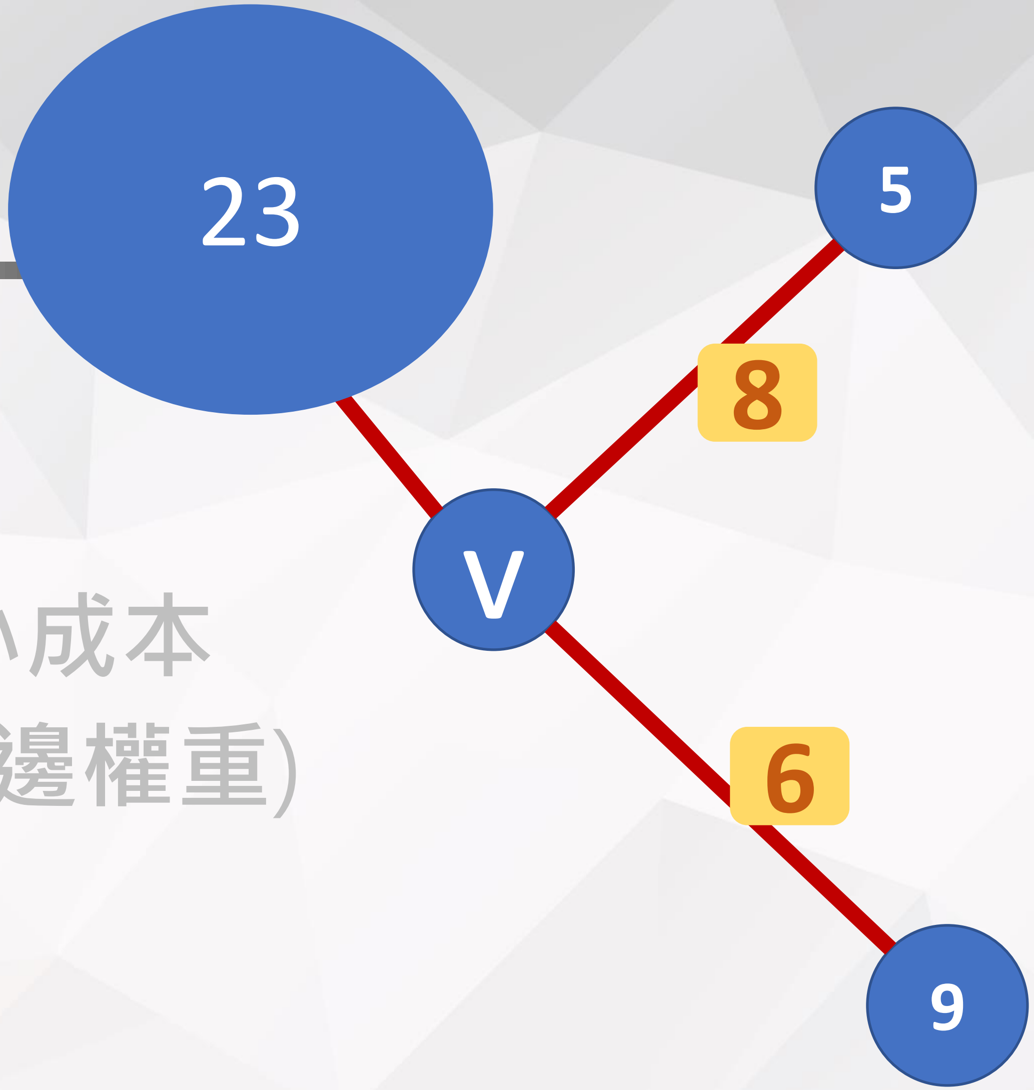

Relaxation

想像自己是圖中的某點 v
身旁有一些鄰點 u
u 知道源點到他們那裡的**最小成本**
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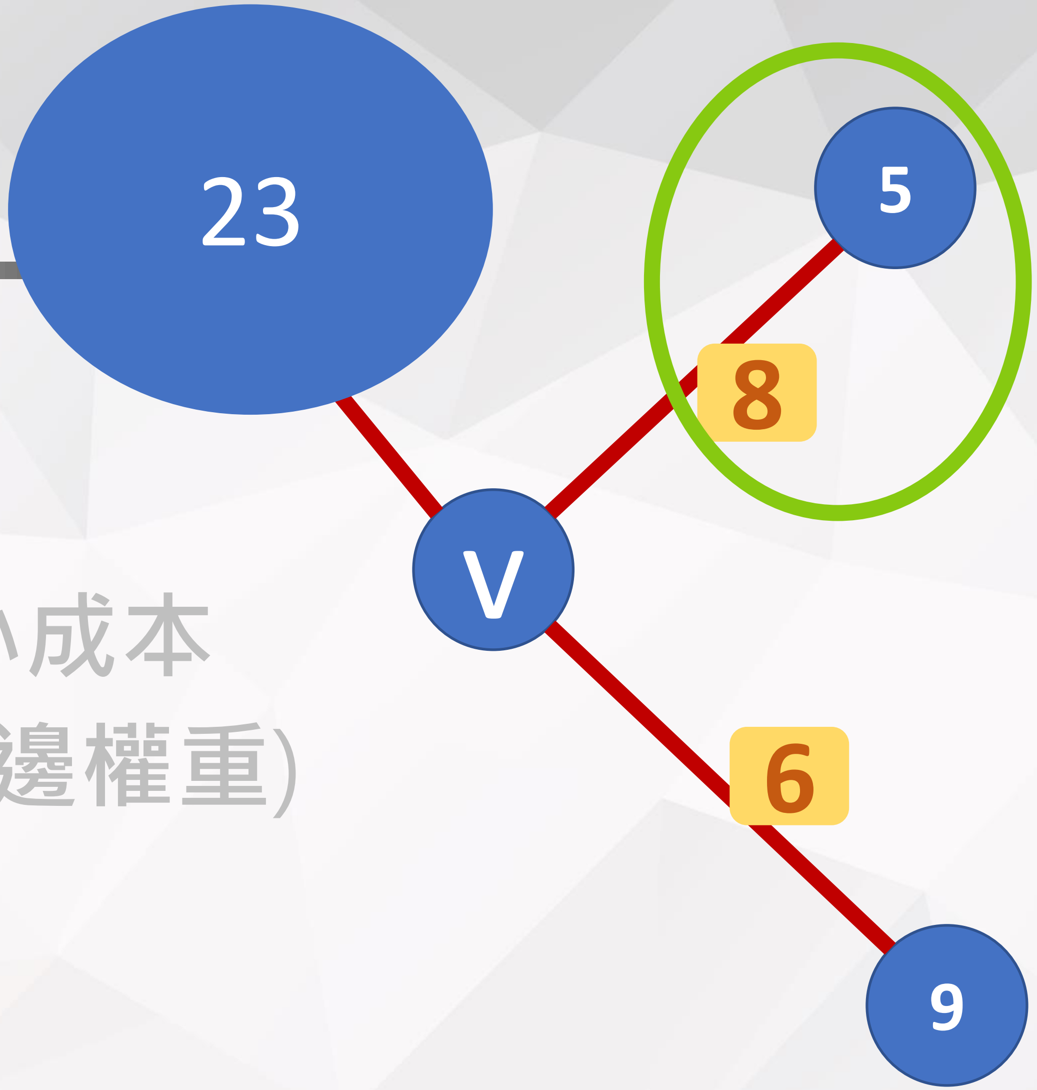

Relaxation

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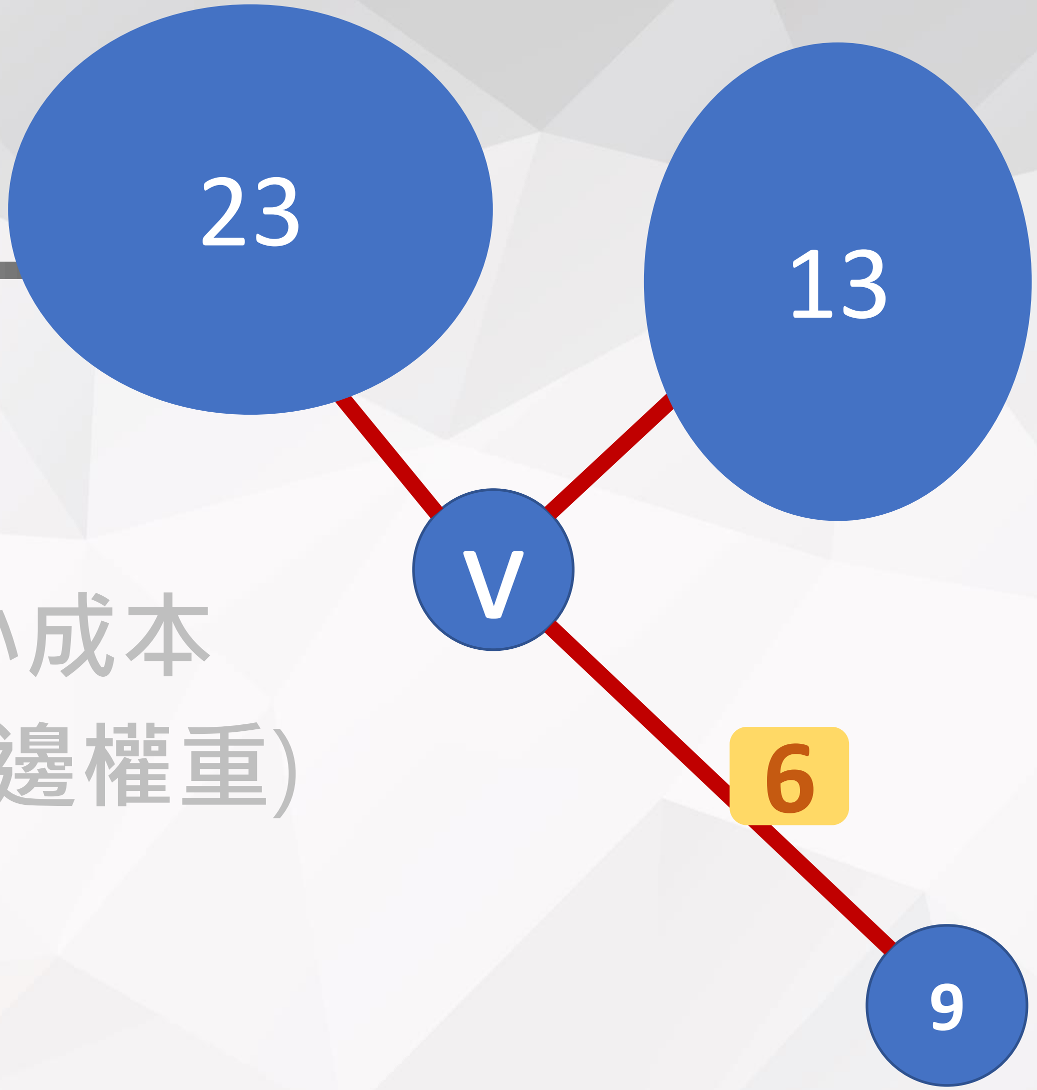

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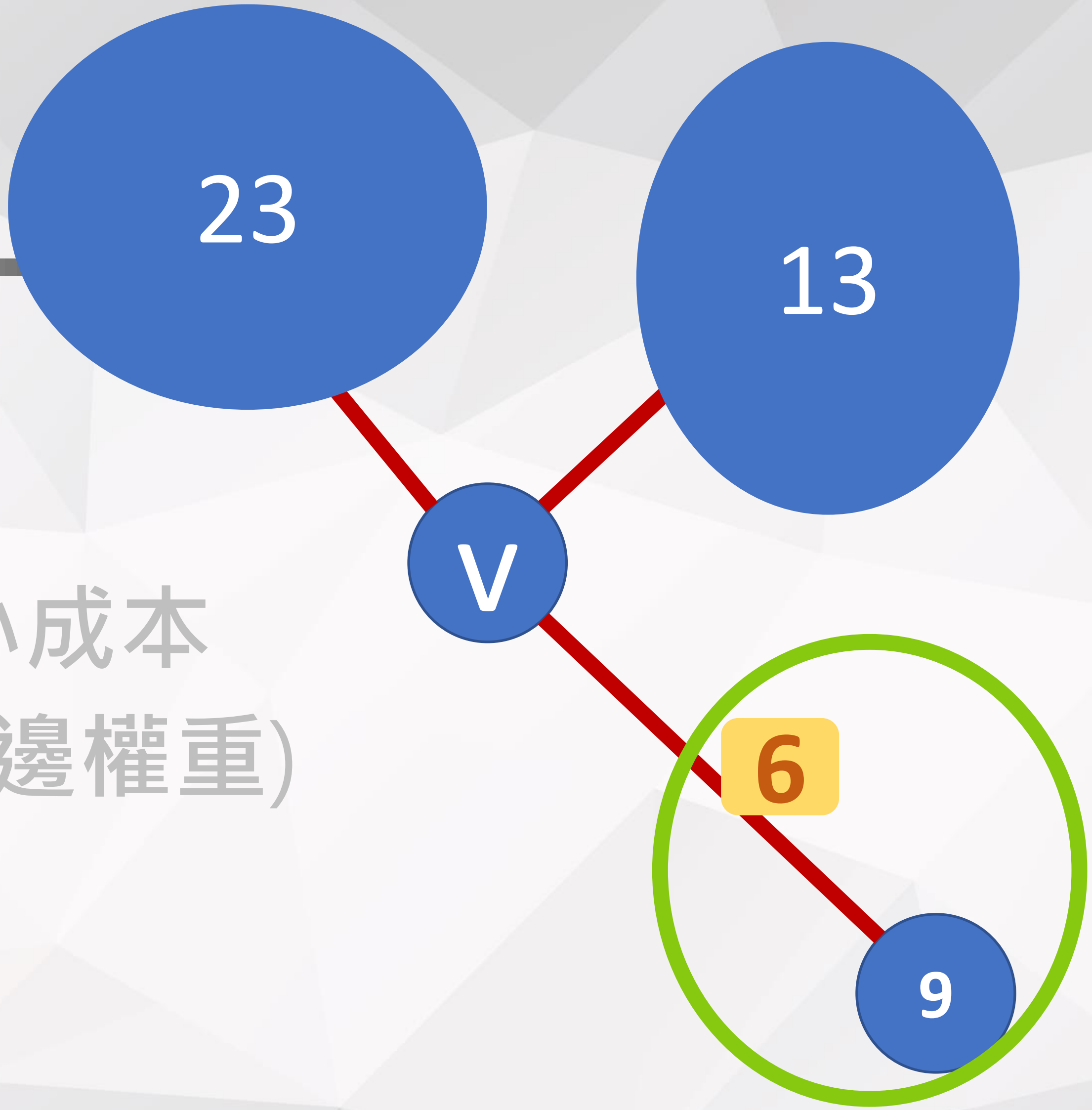

Relaxation

想像自己是圖中的某點 v

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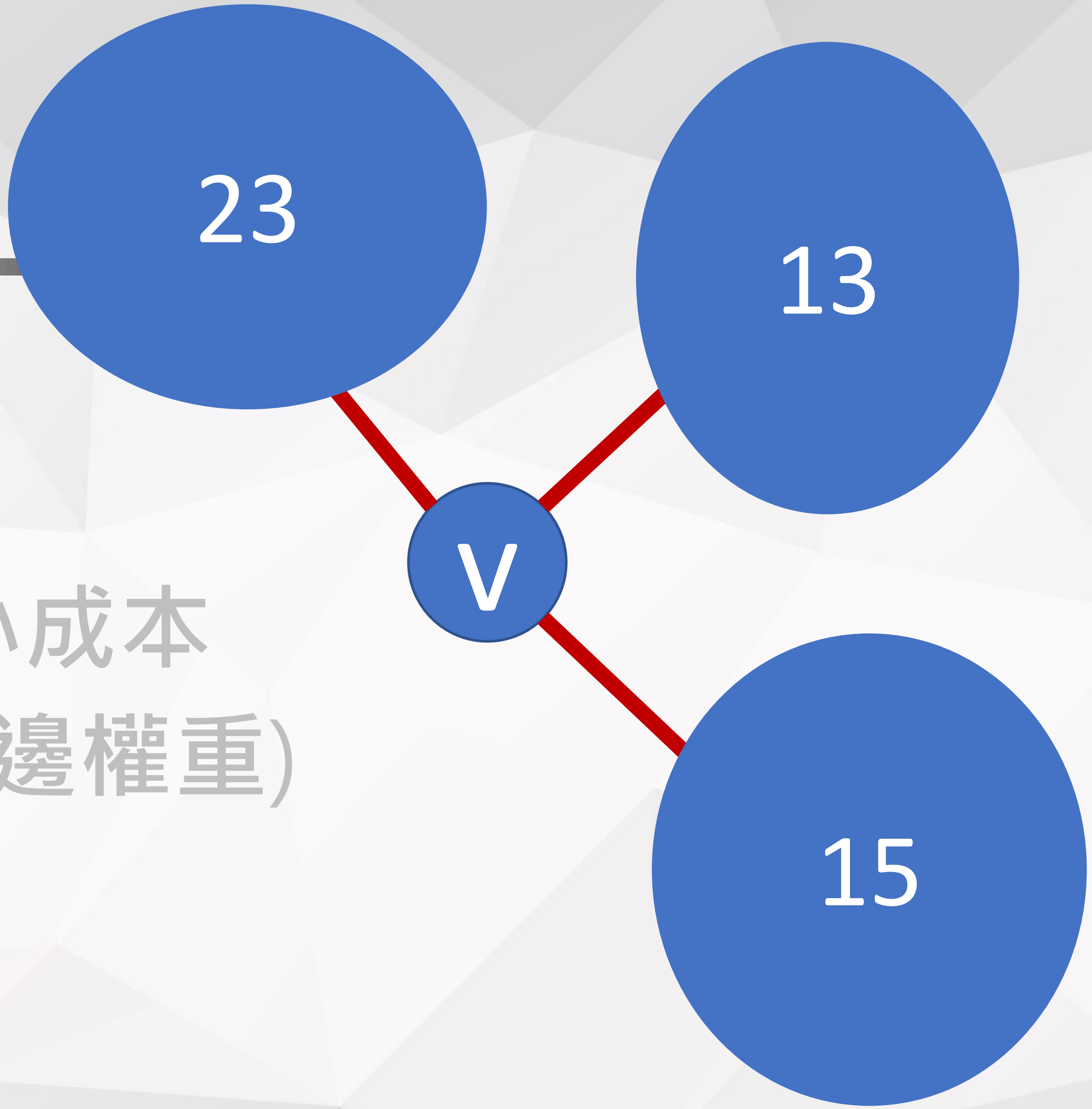

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想像自己是圖中的某點 v

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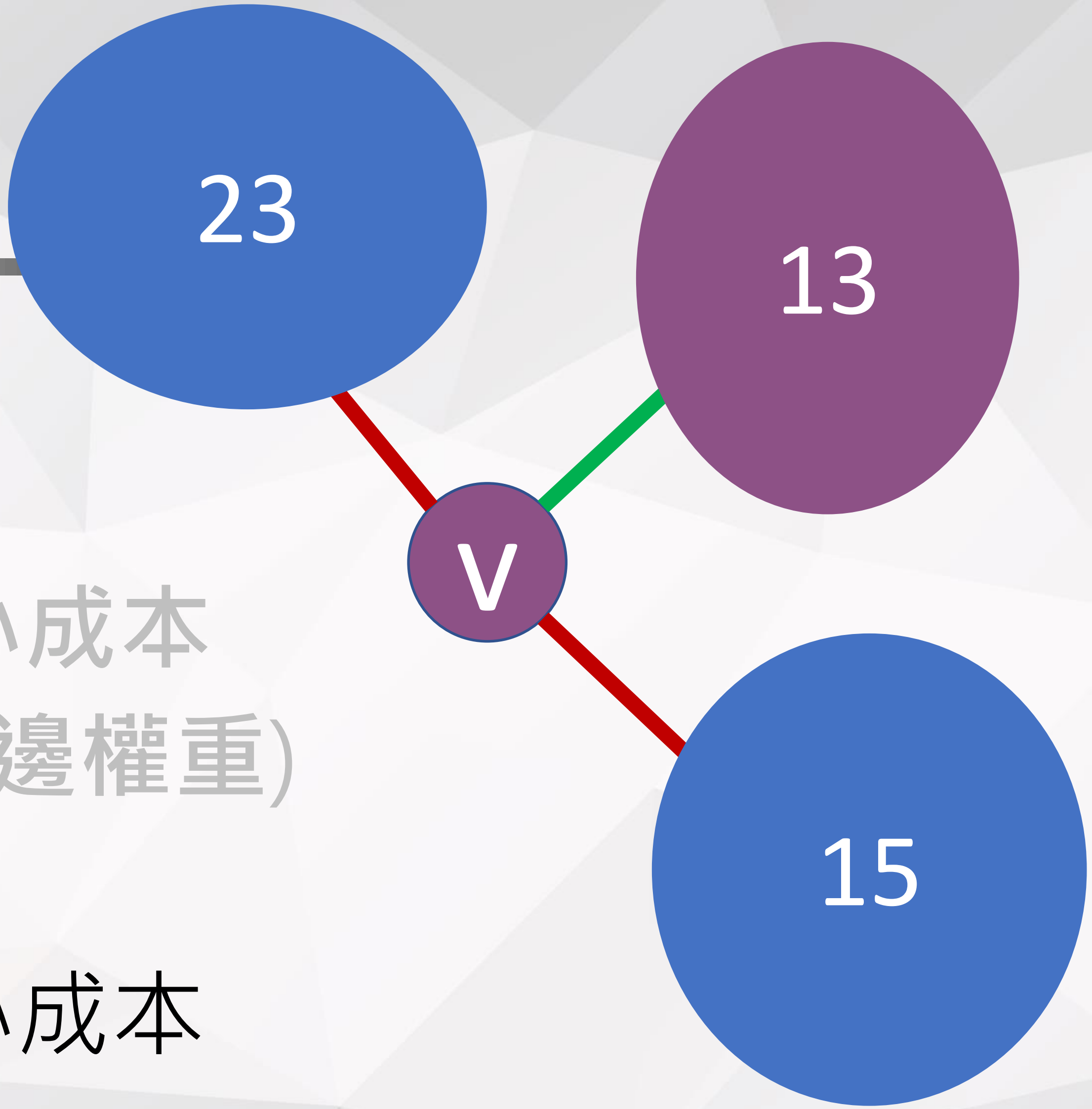
v 就能得知，源點到 v 的最小成本



Relaxation

想像自己是圖中的某點 v

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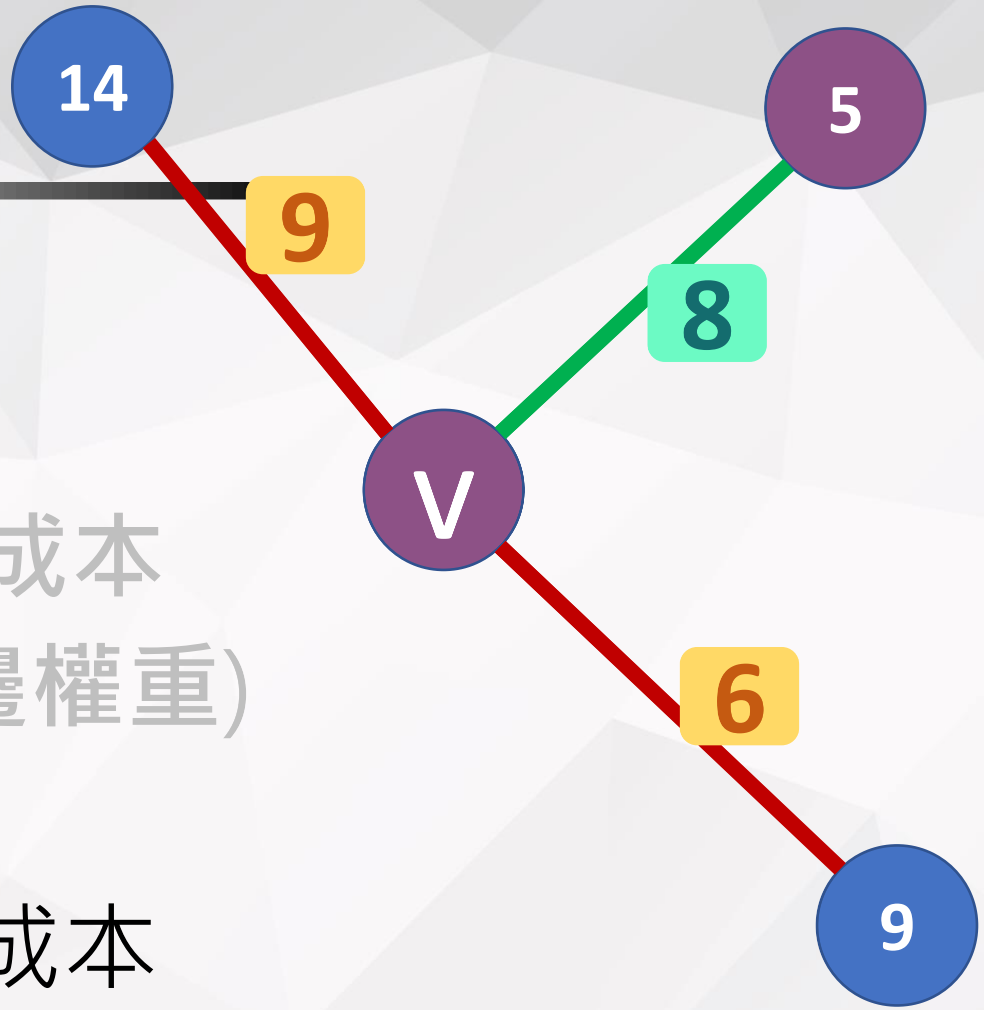

Relaxation

想像自己是圖中的某點 v
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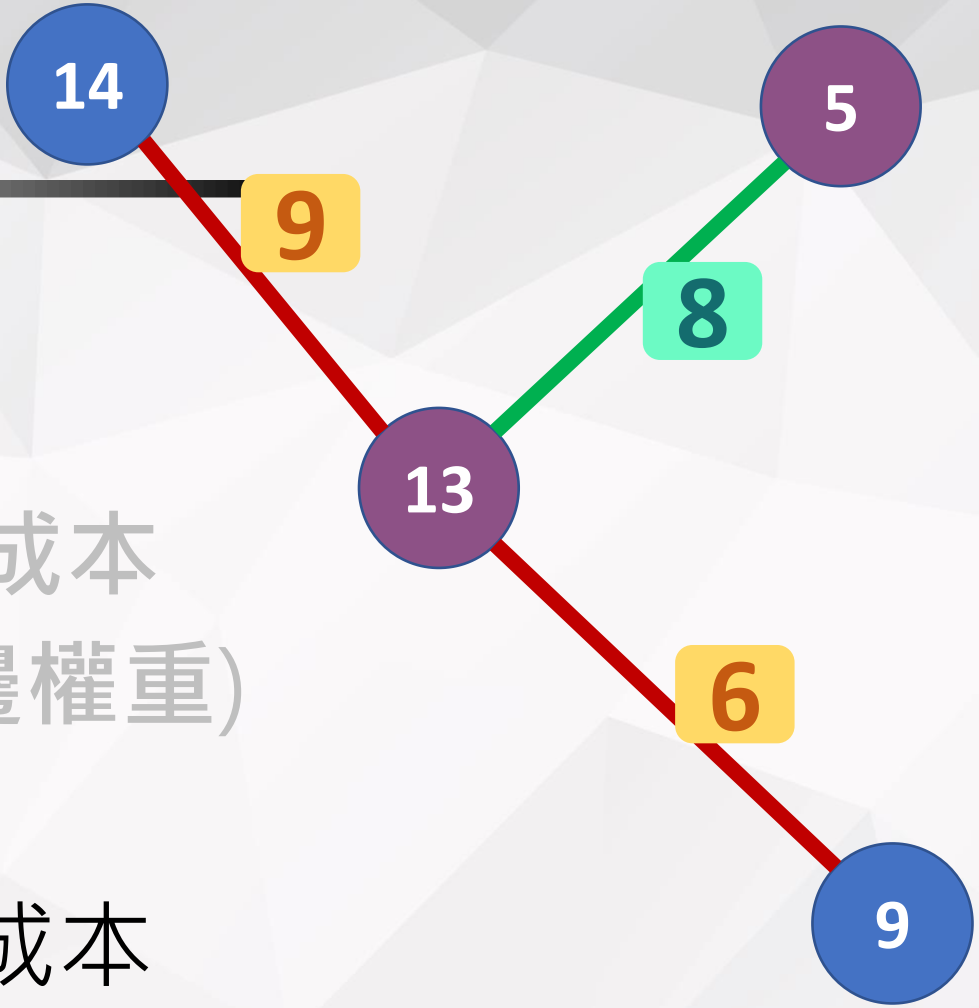


Relaxation 實作

```
int update = cost[u] + w; // w := weight

cost[v] = min(cost[v], update);
```

Questions?

單源最短路徑

- Relaxation
- Dijkstra's algorithm
- Bellman-Ford's algorithm

Dijkstra's algorithm

Dijkstra 實作

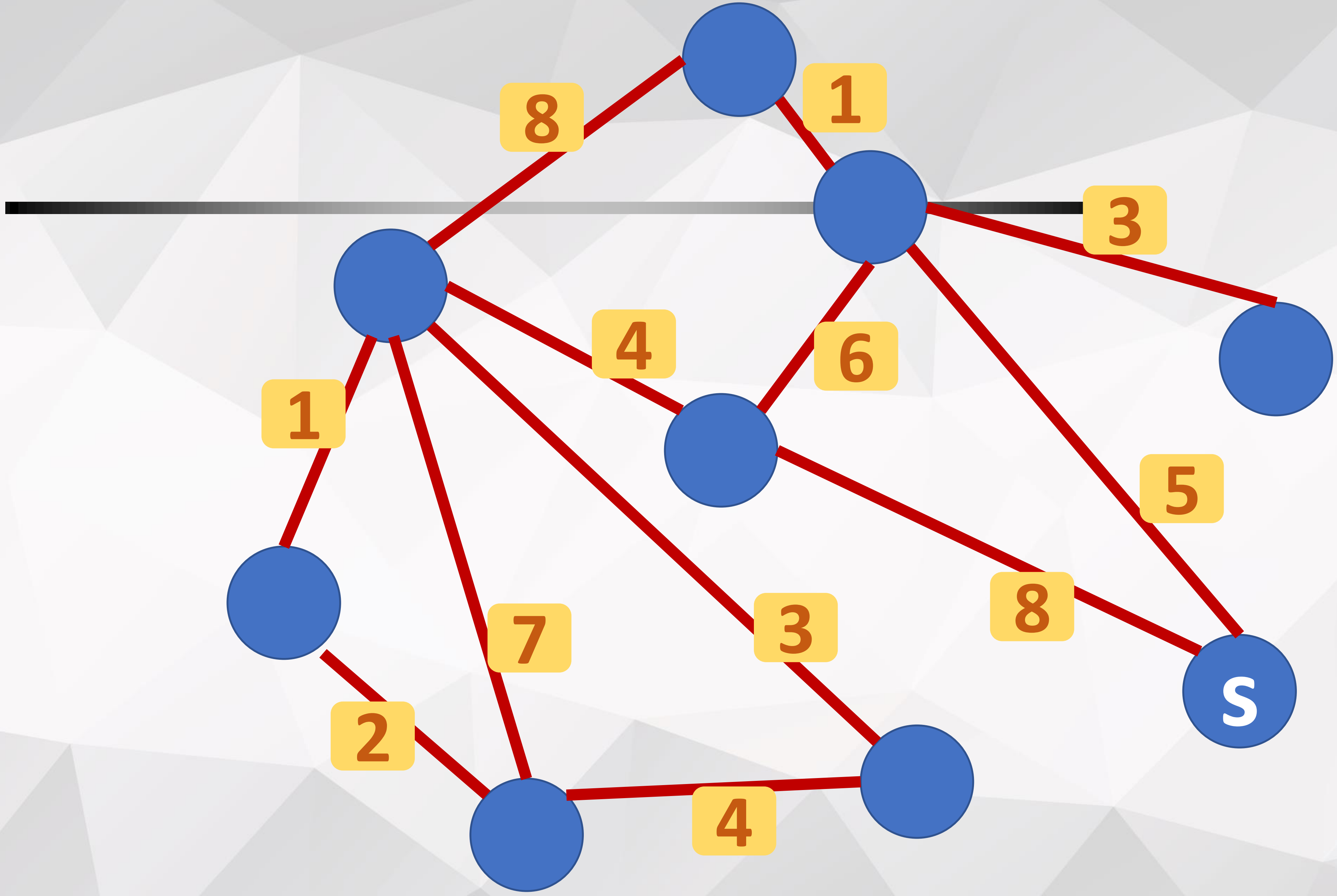
```
vector<node> E[maxn]; // 邊集合
      ⋮

/* 假設輸入完邊的資訊了 */
```

Dijkstra 實作 (**初始化**)

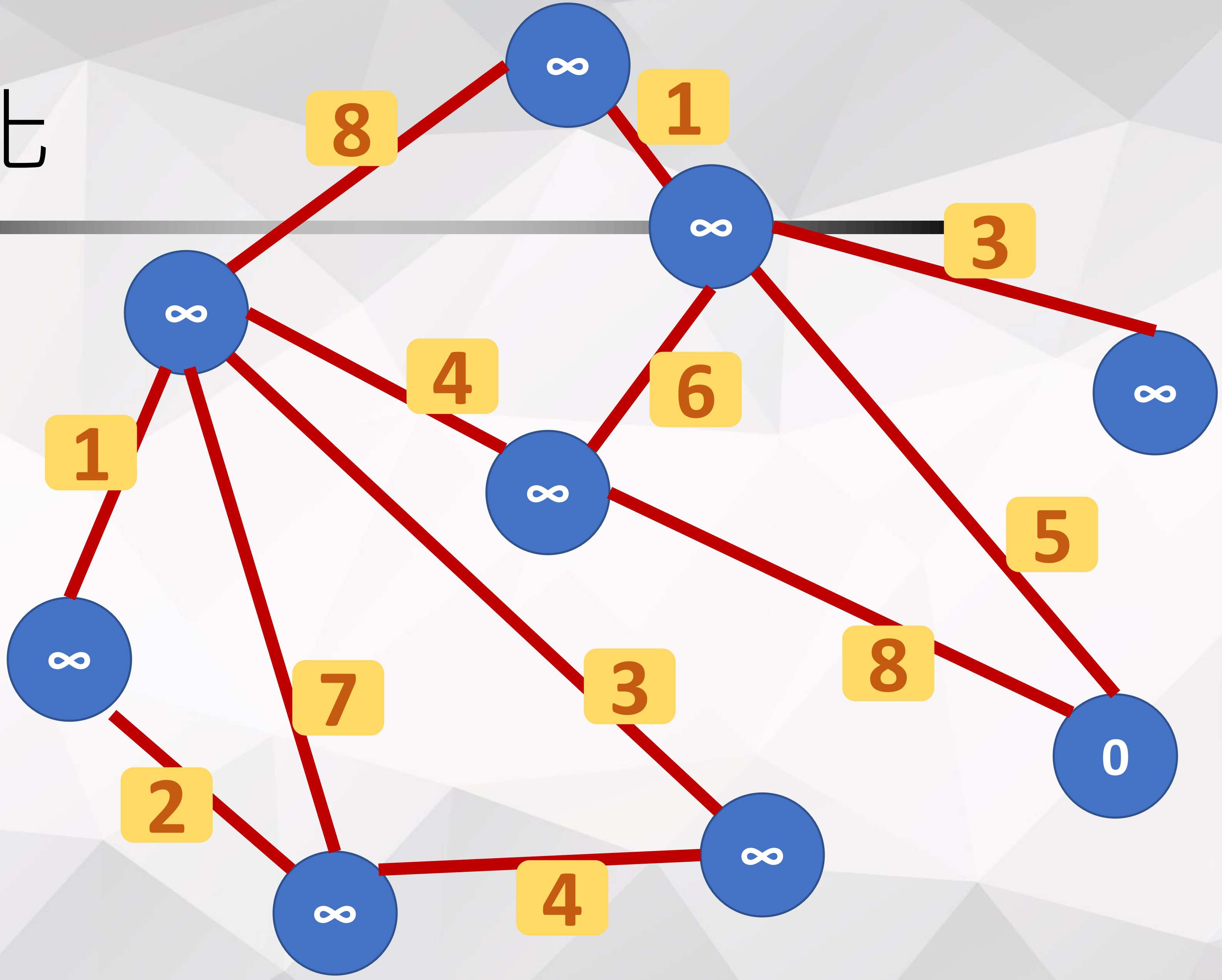
```
memset(s, 0x3f, sizeof(s)); //初始為無限大

priority_queue<node> Q;

Q.push({source, s[source] = 0});
```

Dijkstra 實作

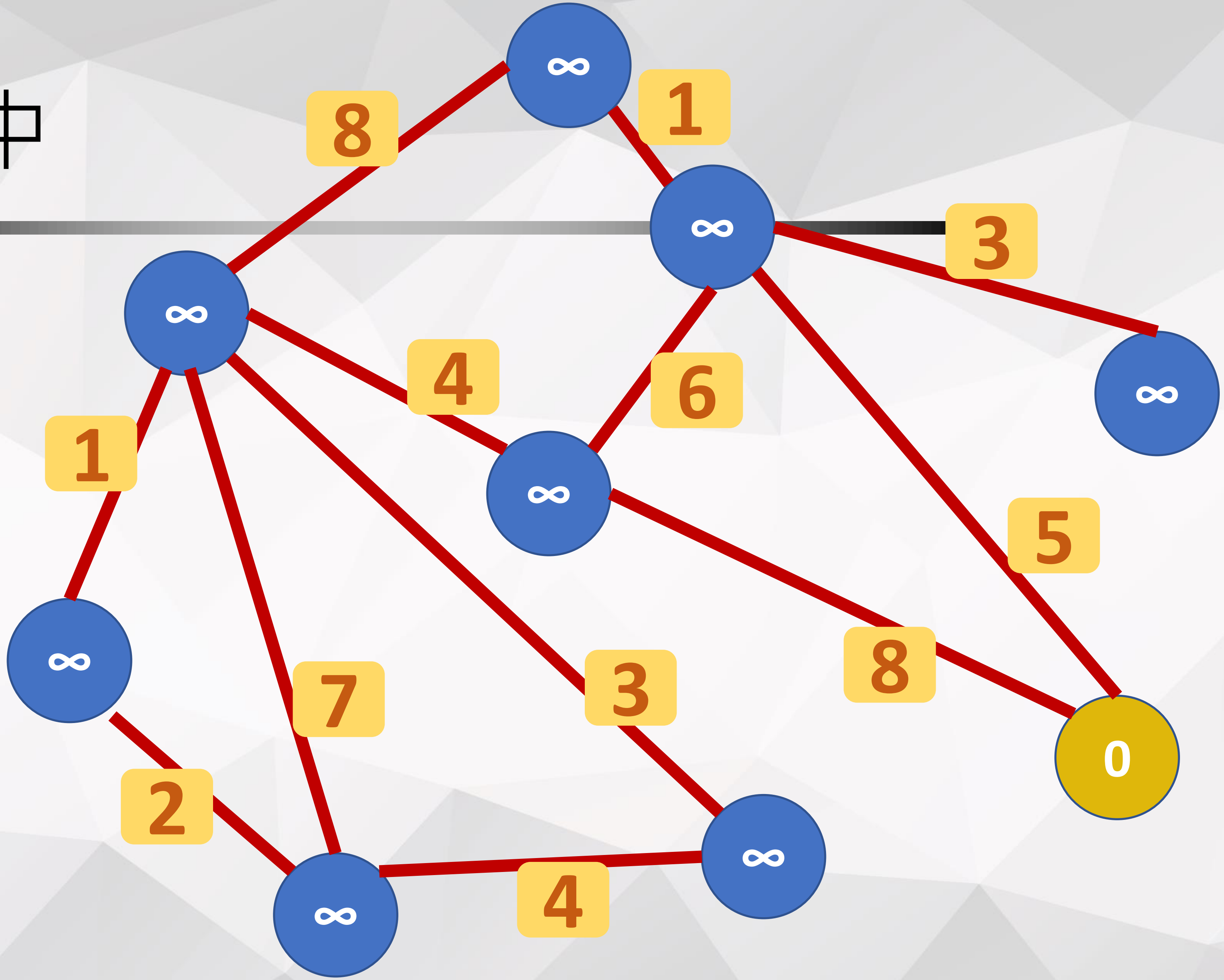
```
while (!Q.empty()) {
    node u = Q.top(); Q.pop();

    for (node v: E[u.id]) {
        int update = u.w + v.w;

        if (update < s[v.id])
            Q.push({v.id, s[v.id] = update});
    }
}
```


8
1
3
4
6
1
5
7
3
8
S
2
4

初始化



拜訪中



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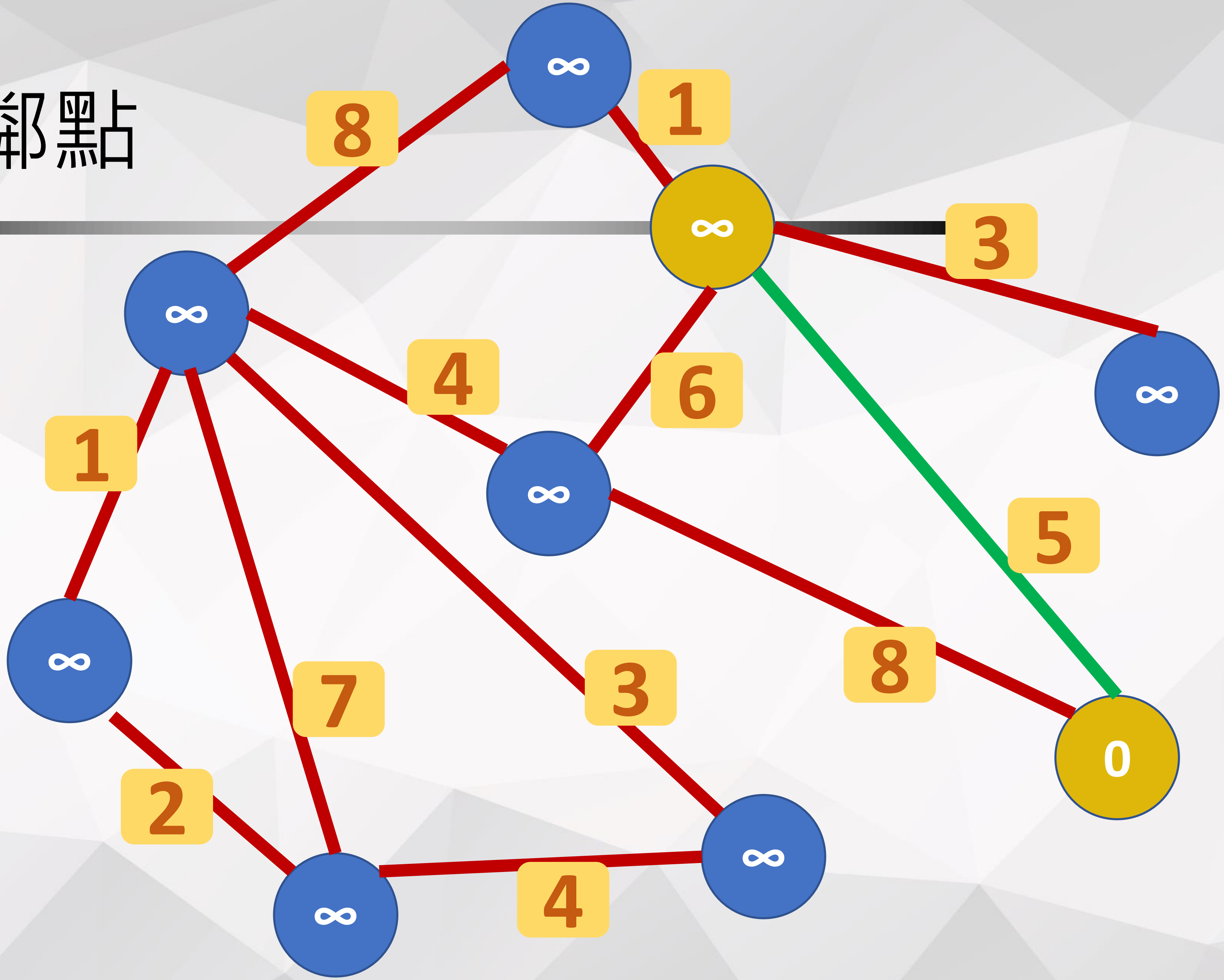

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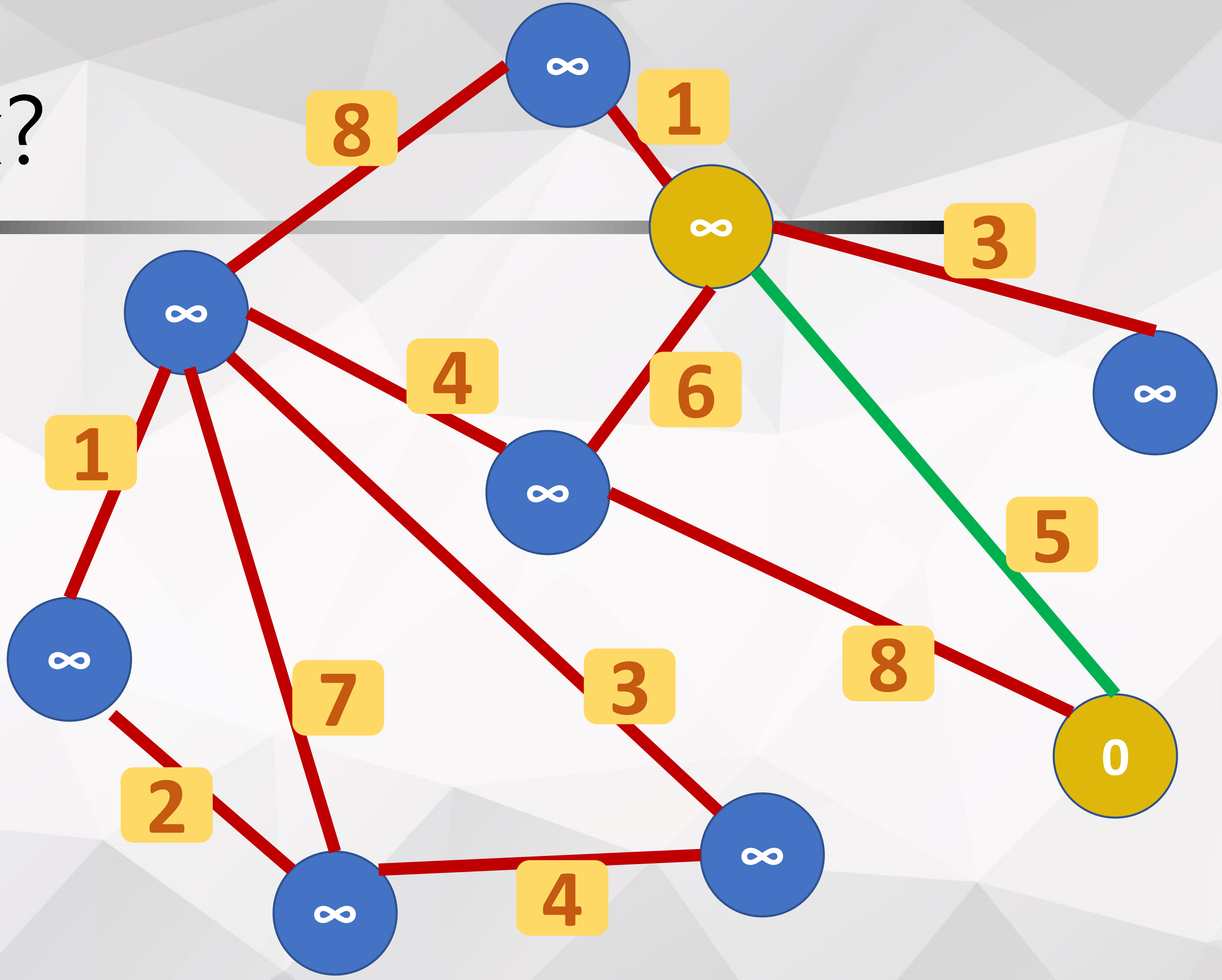
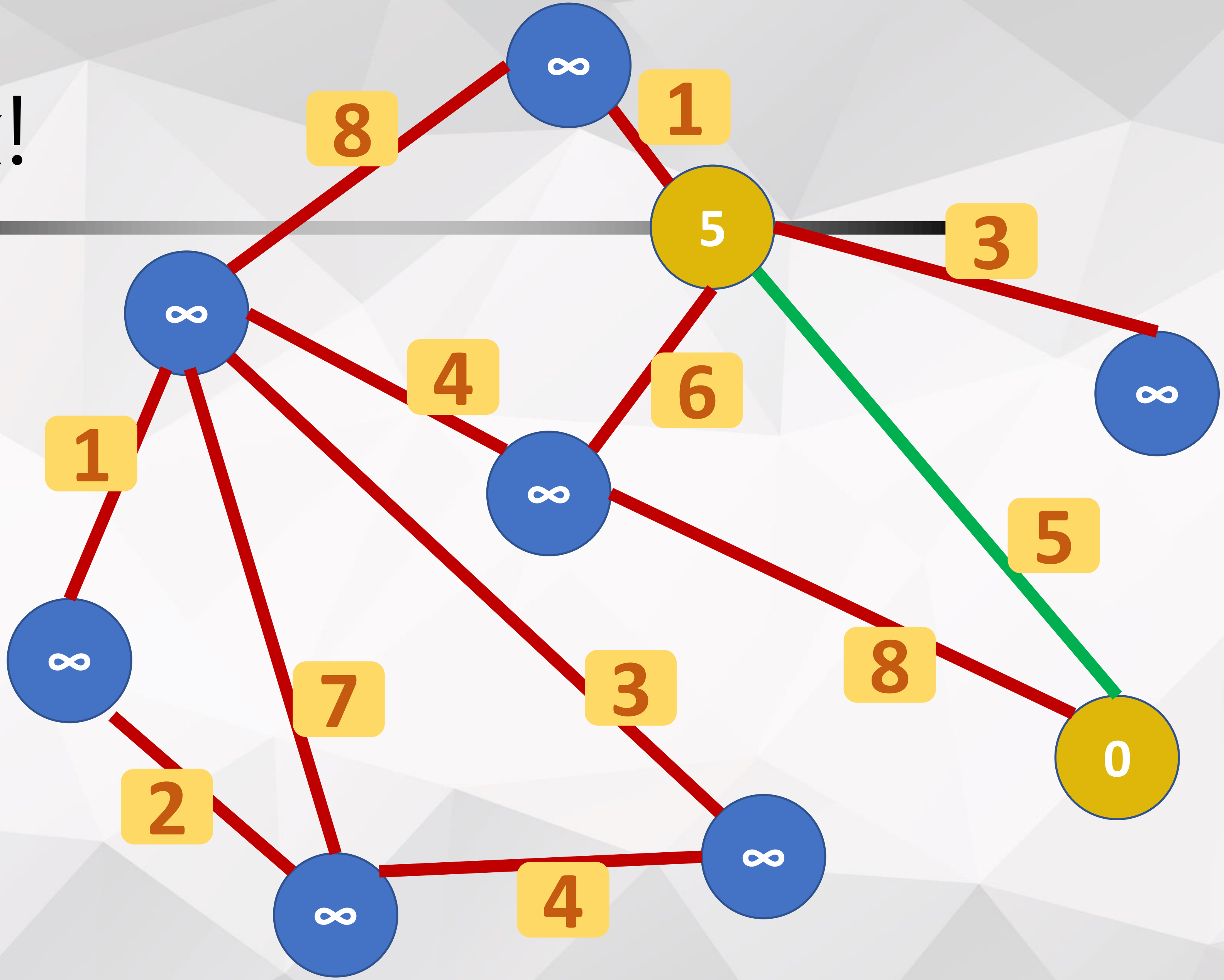


Relax!



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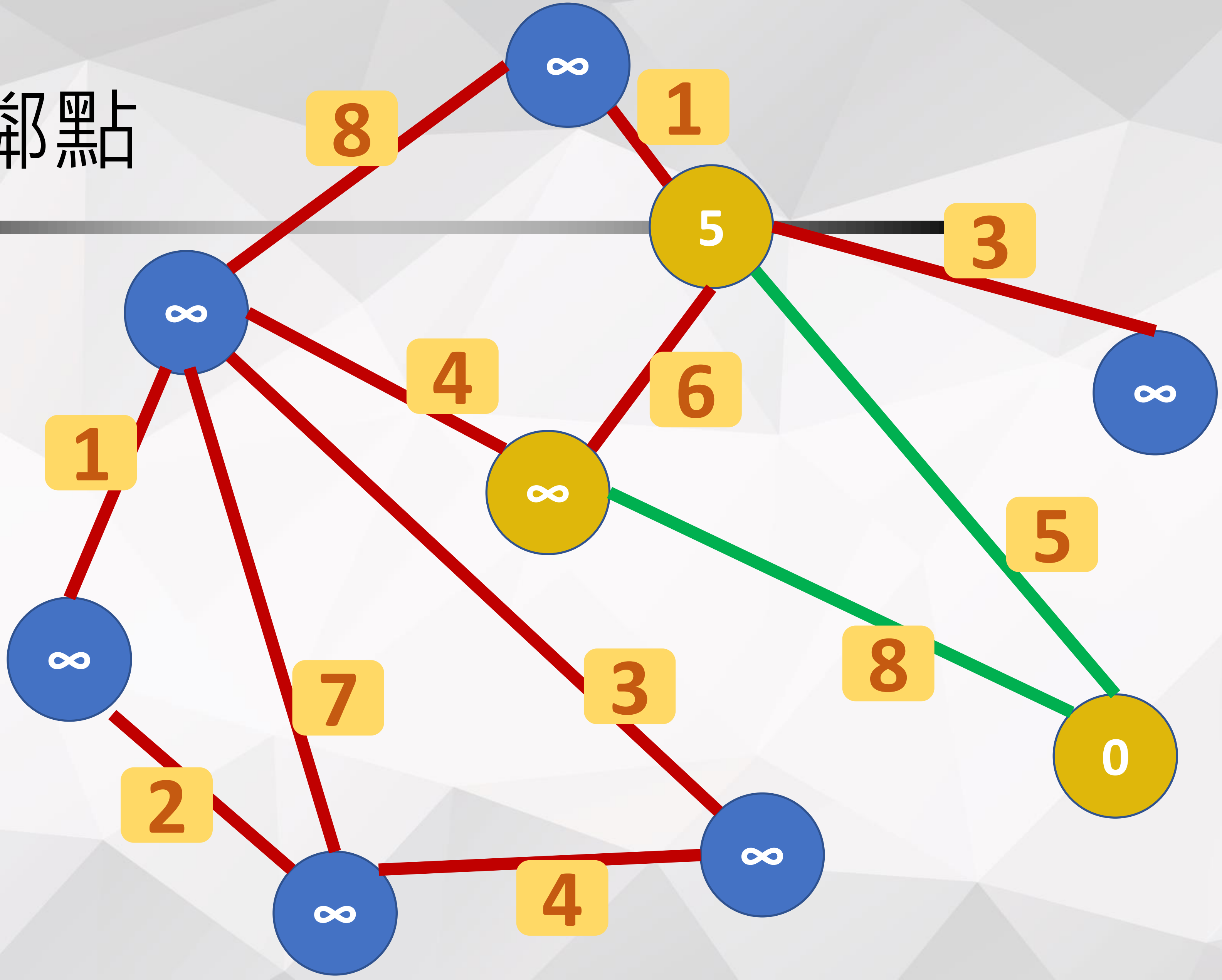

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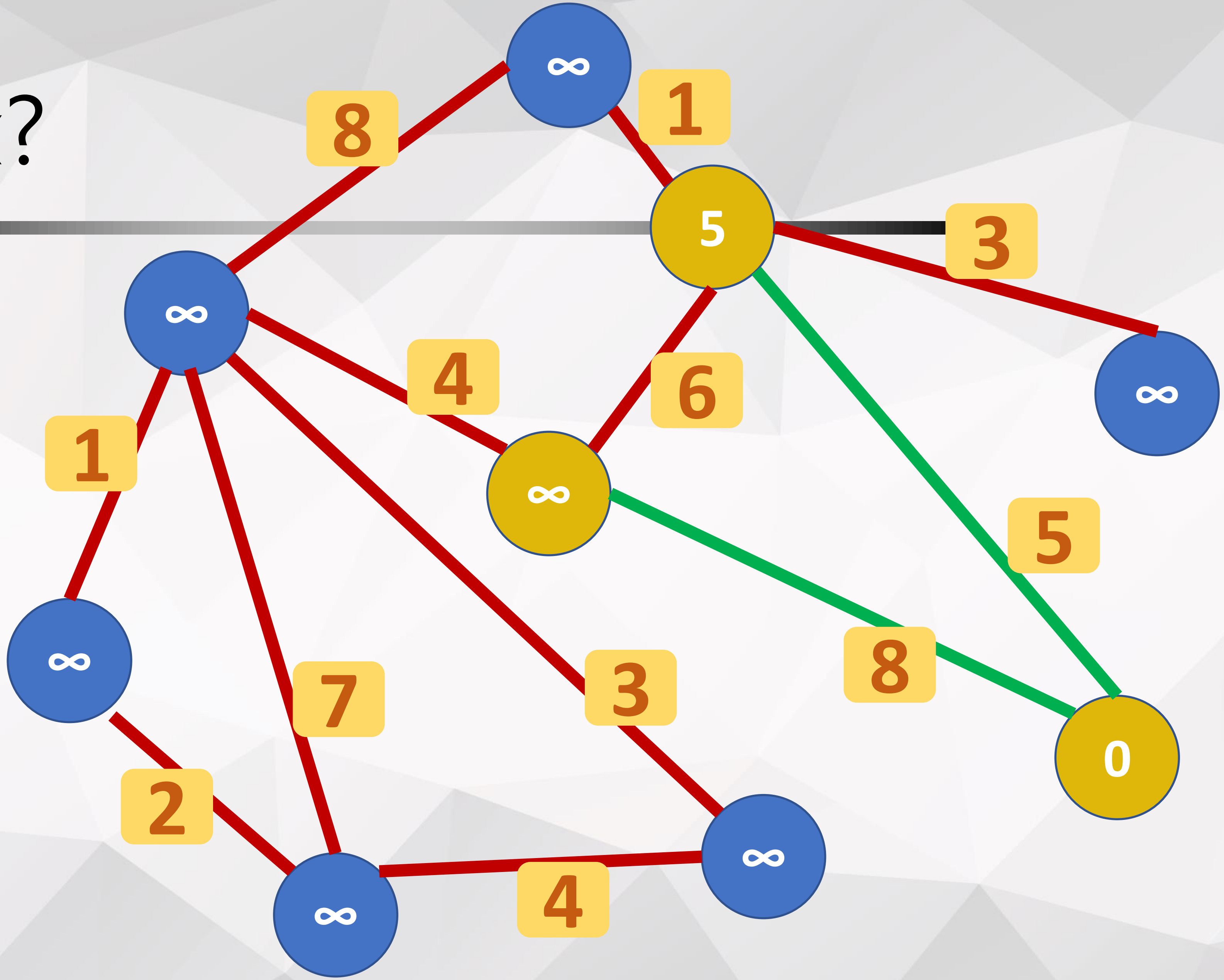

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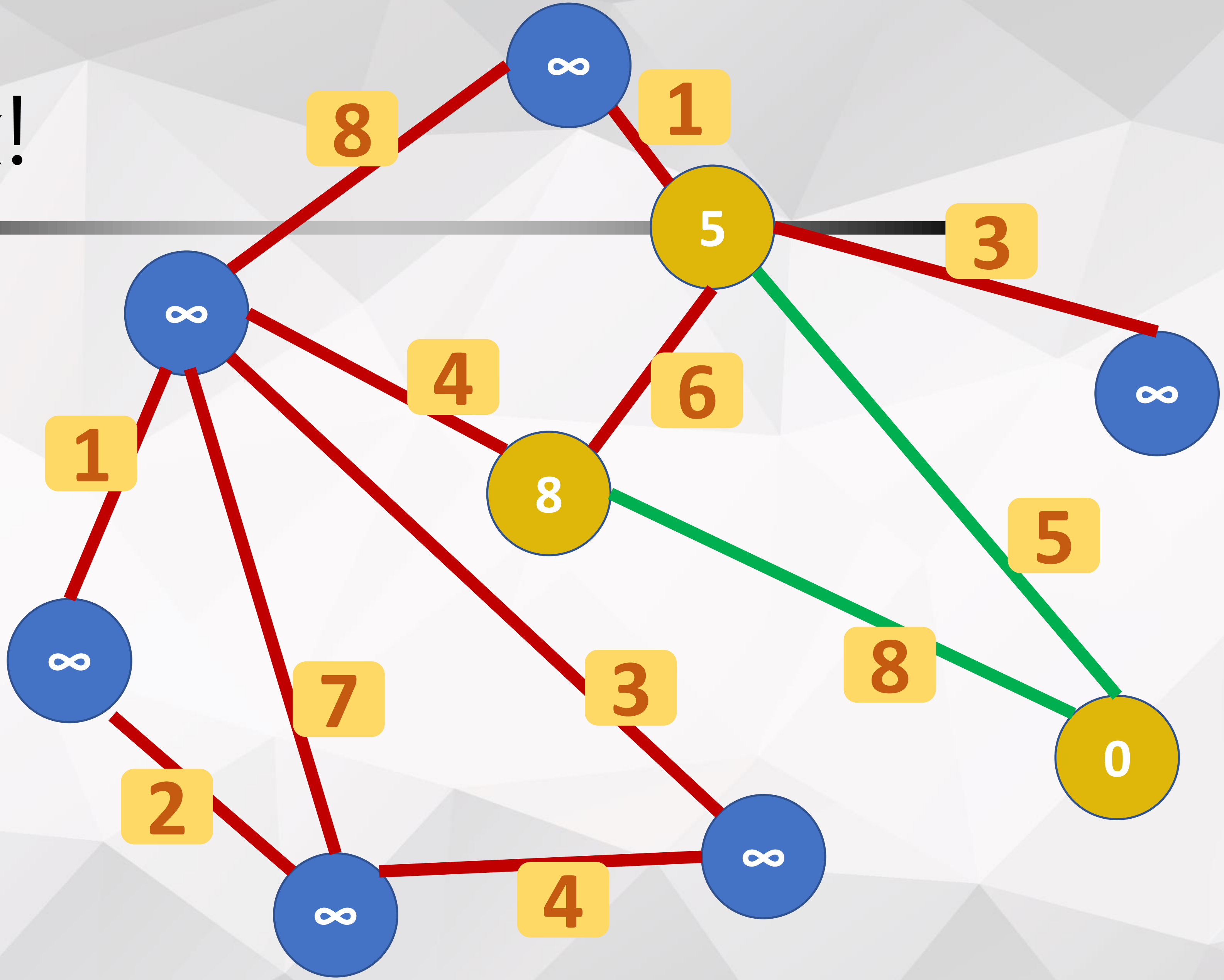
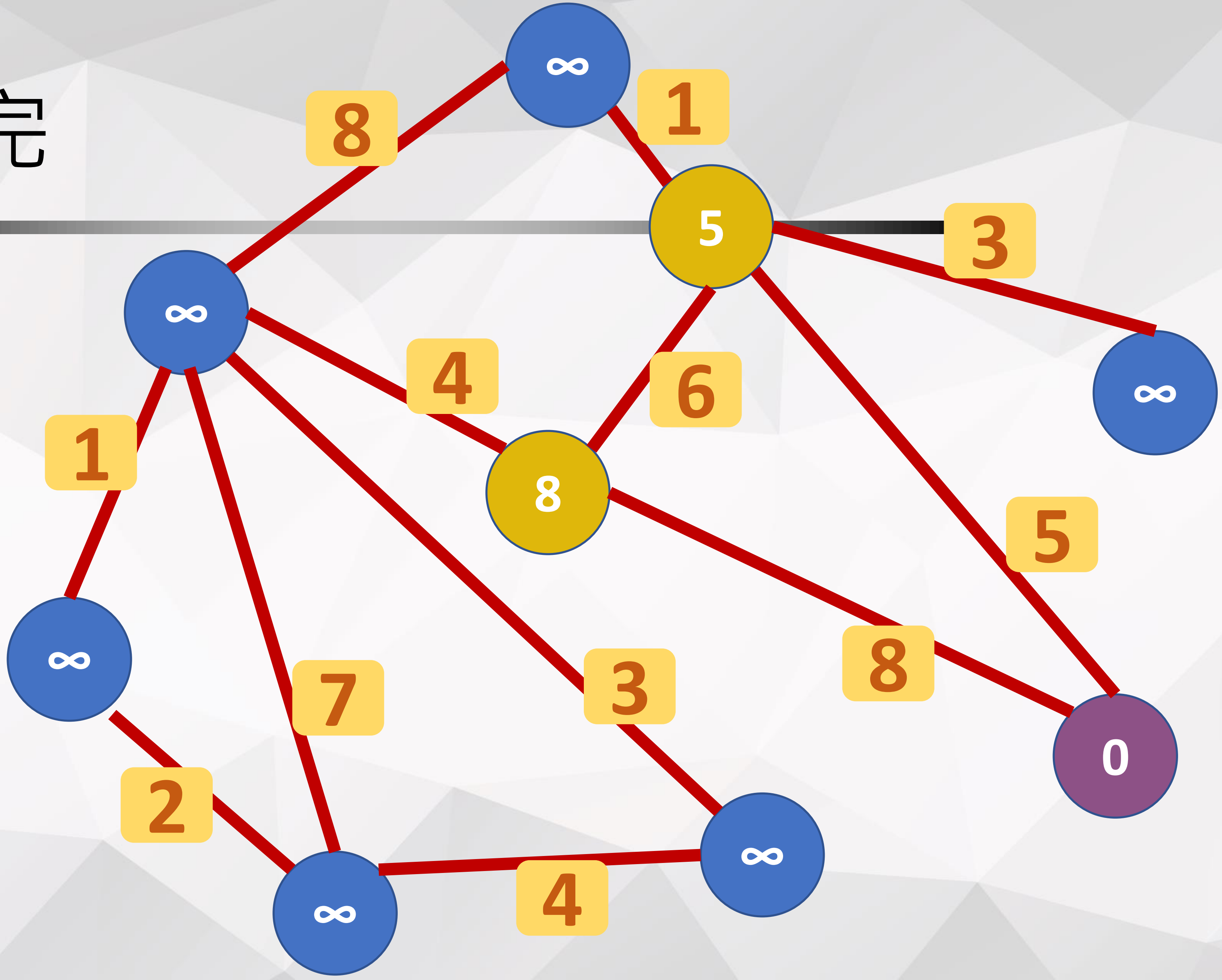
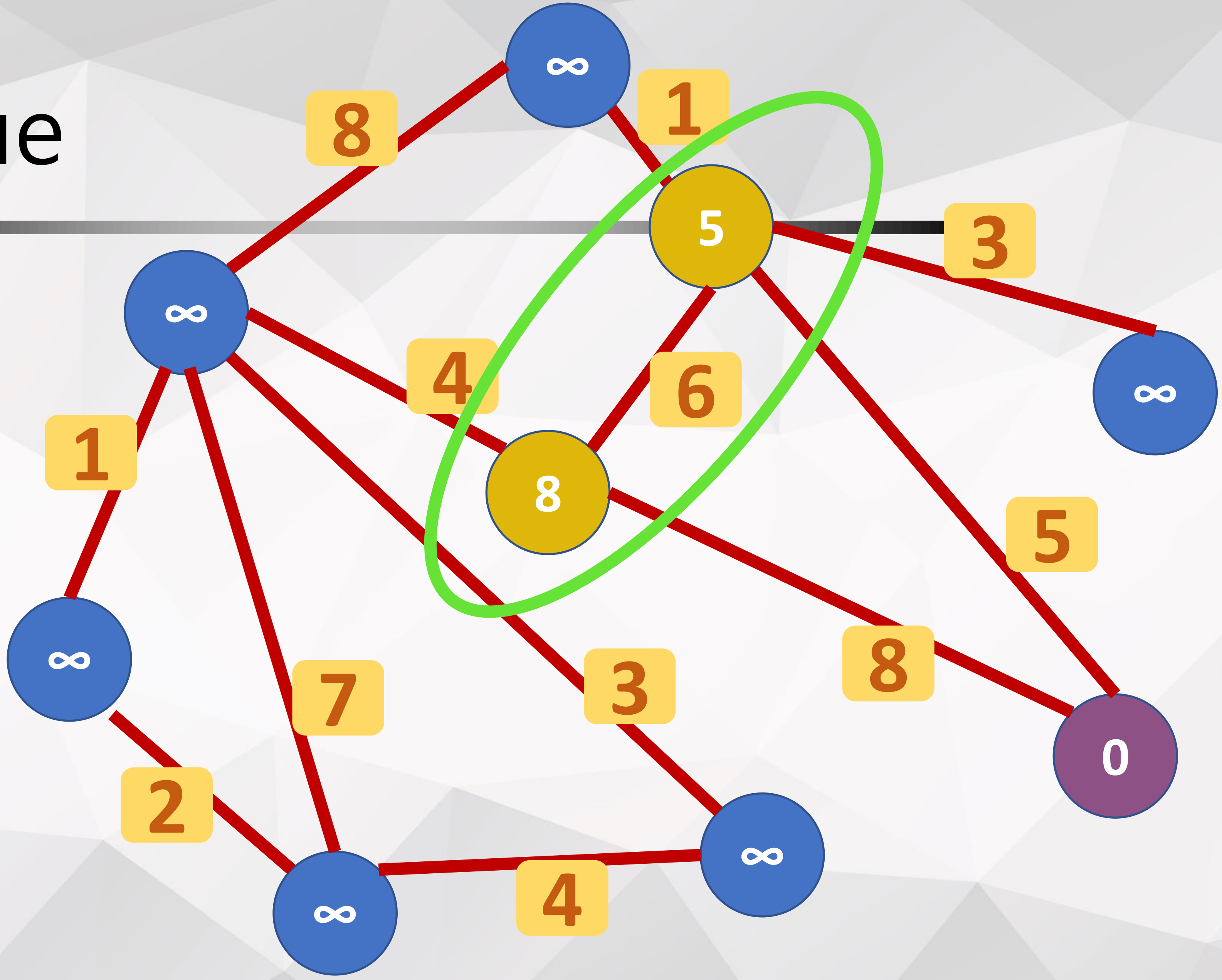


拜訪完



Queue



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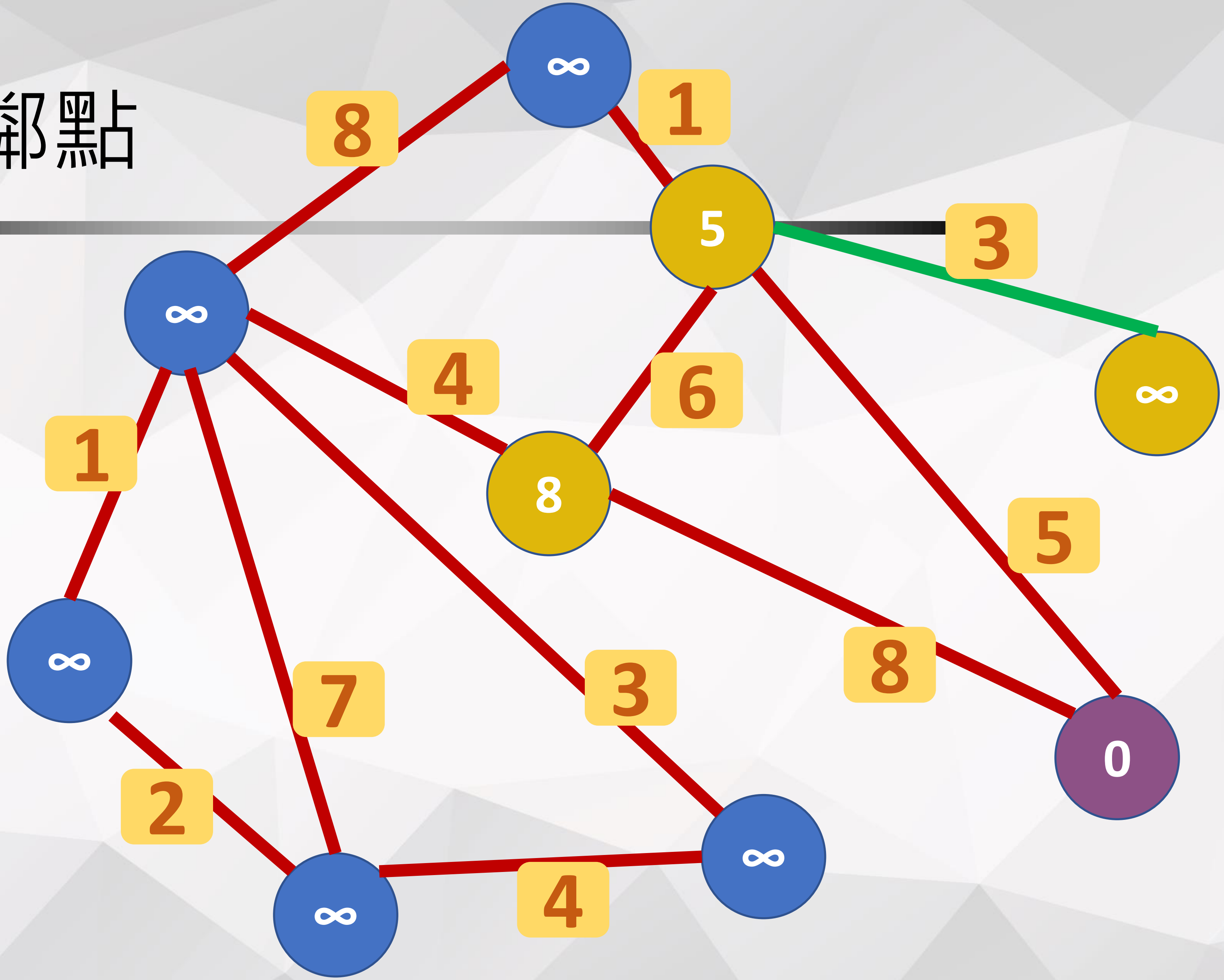
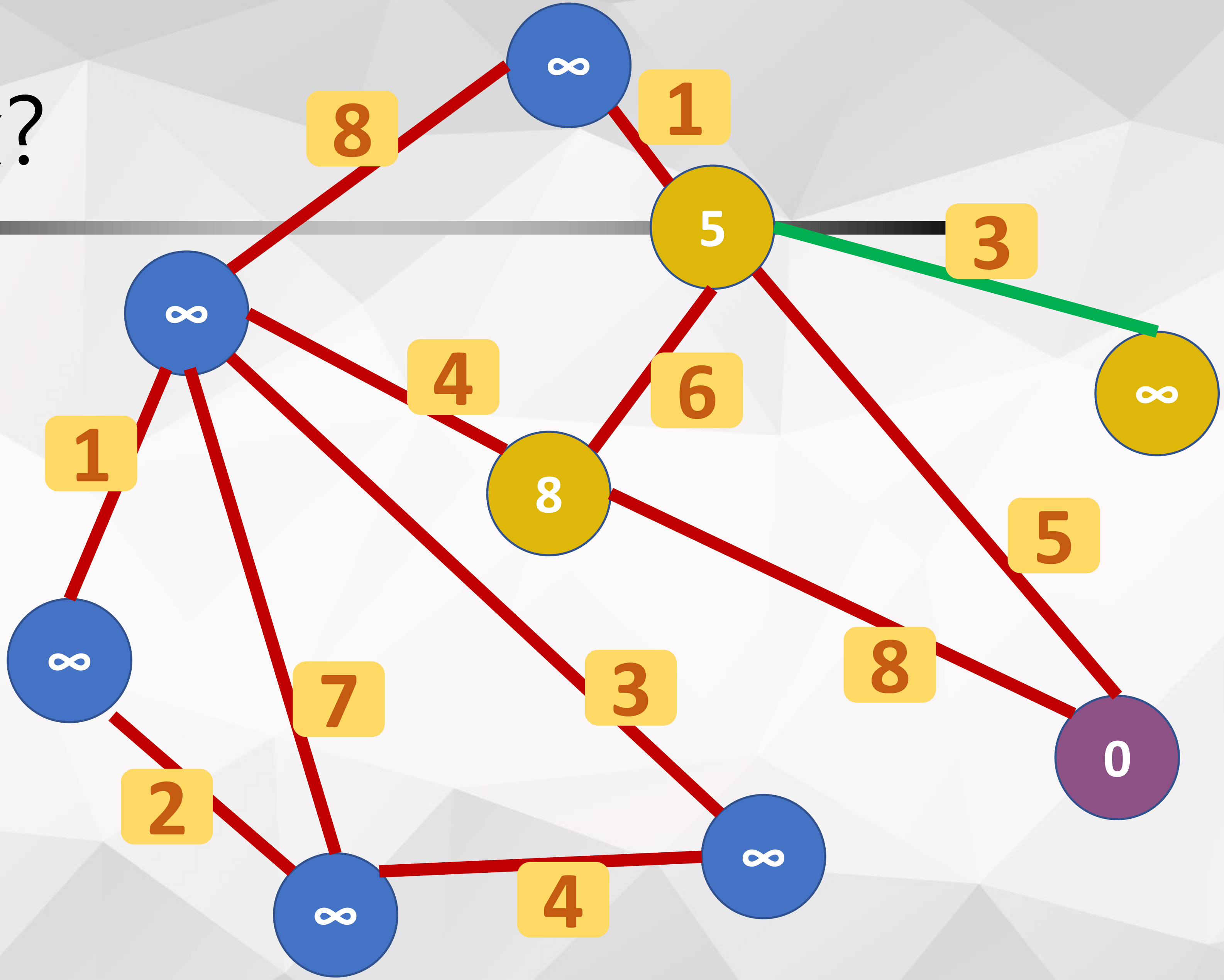


Competitive Programming

Made by 培訓團隊群 CCNS

Relax?



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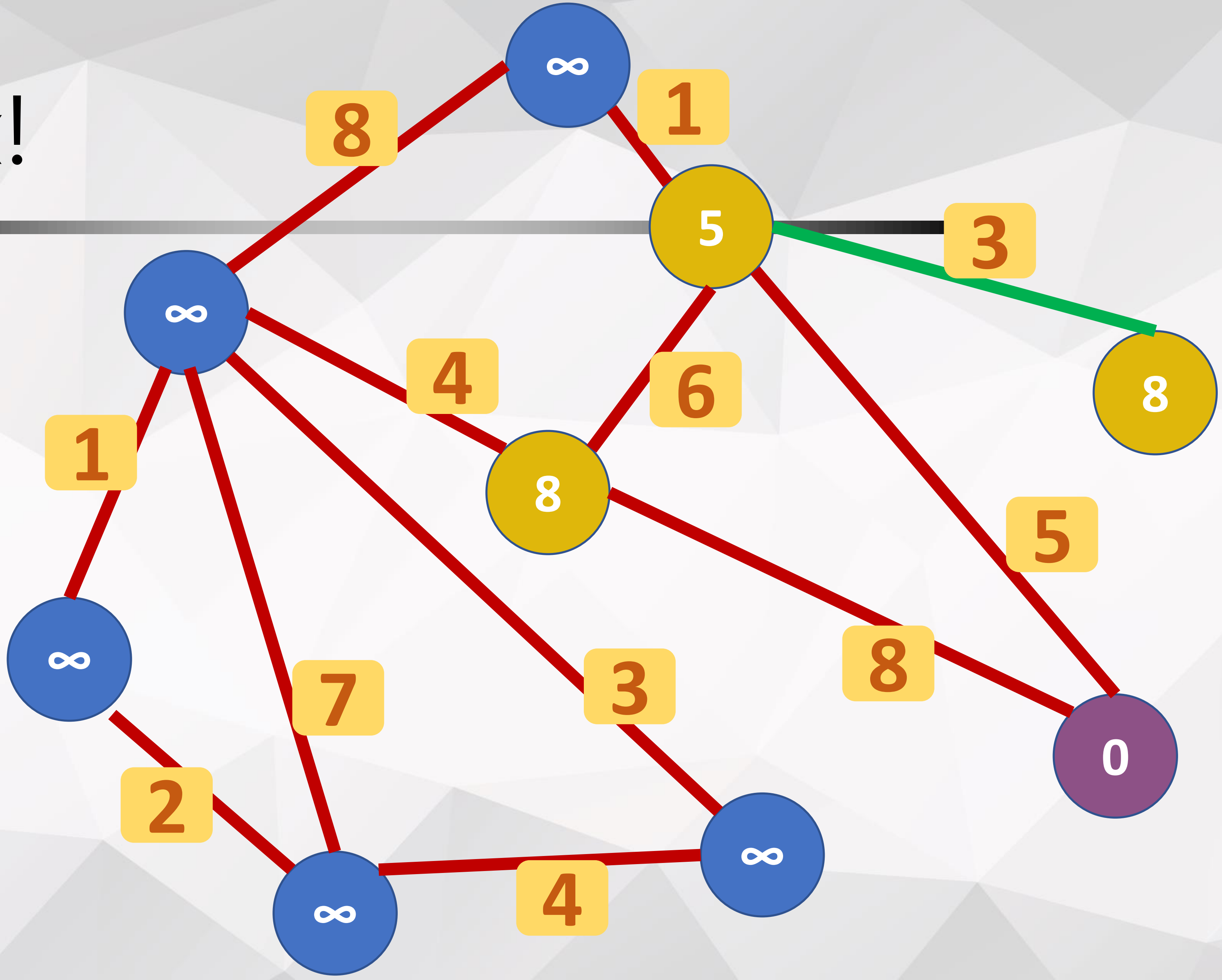

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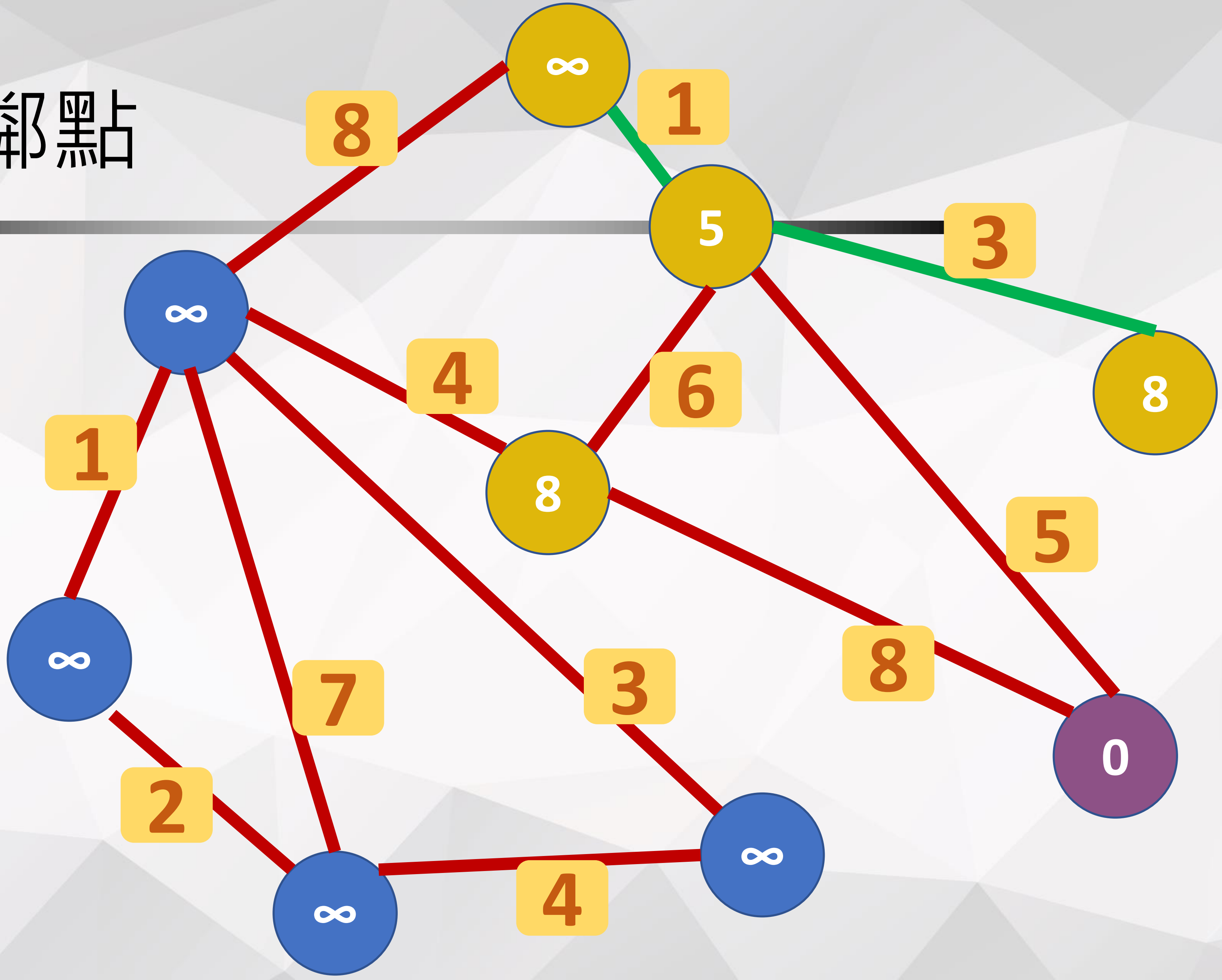

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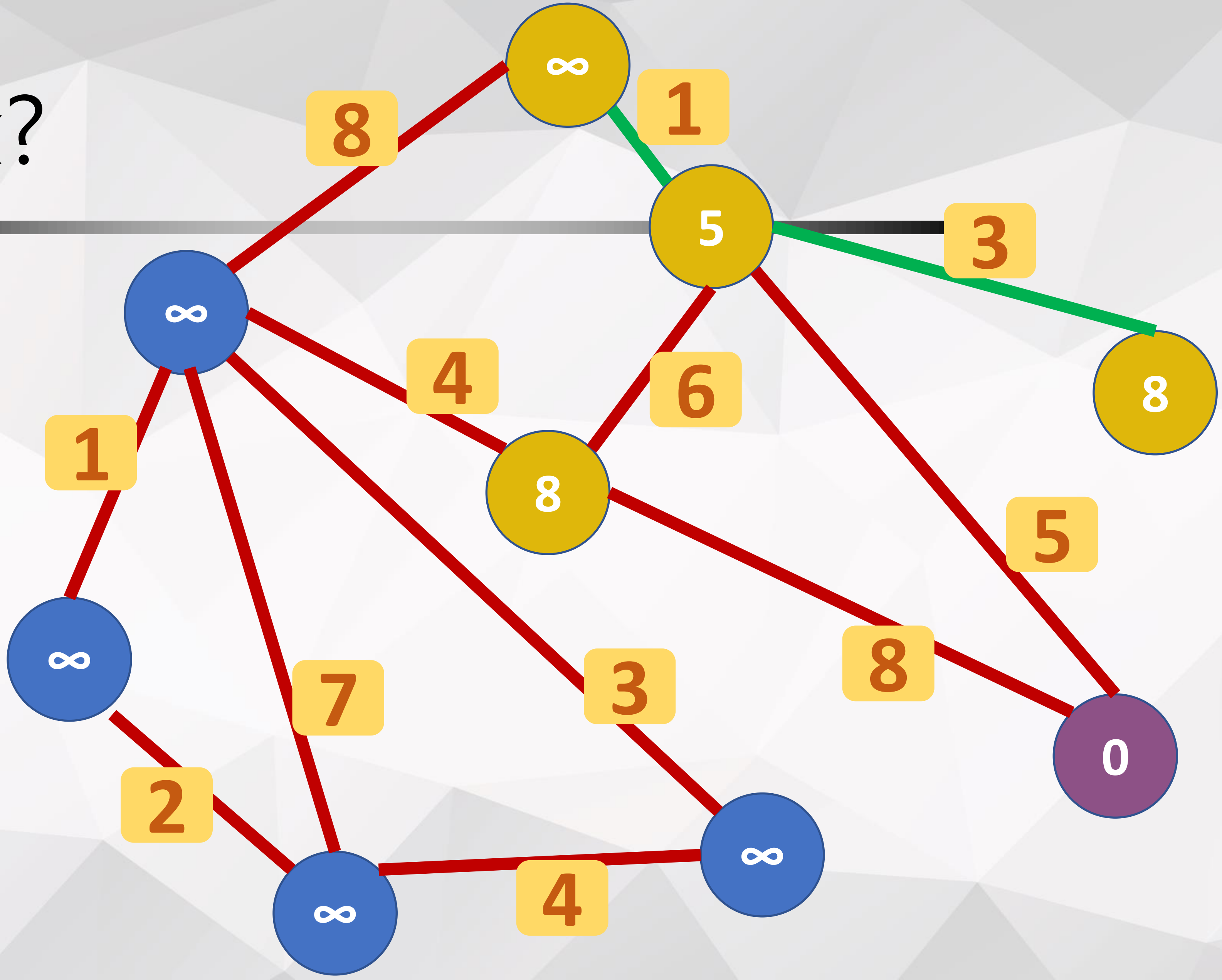

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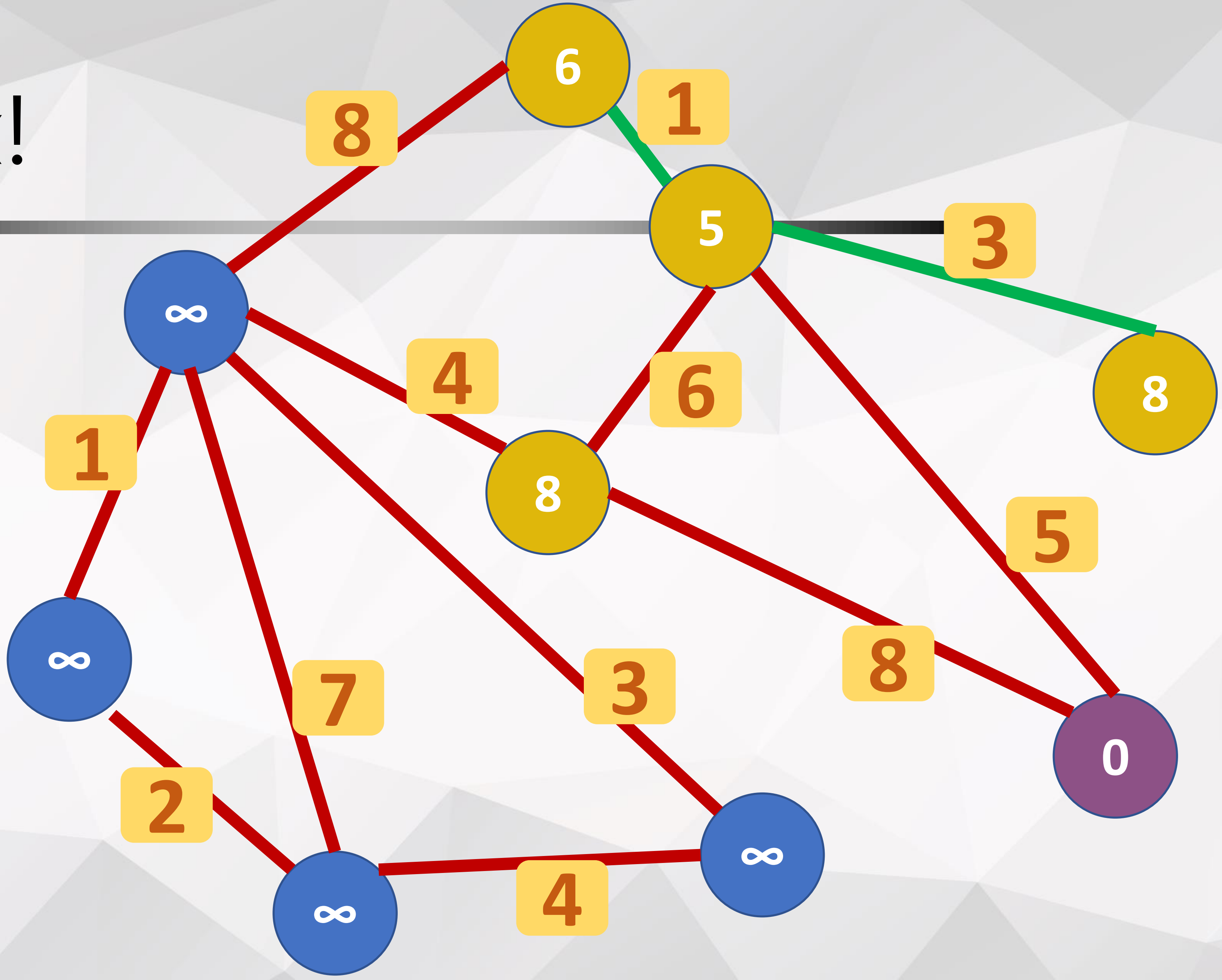

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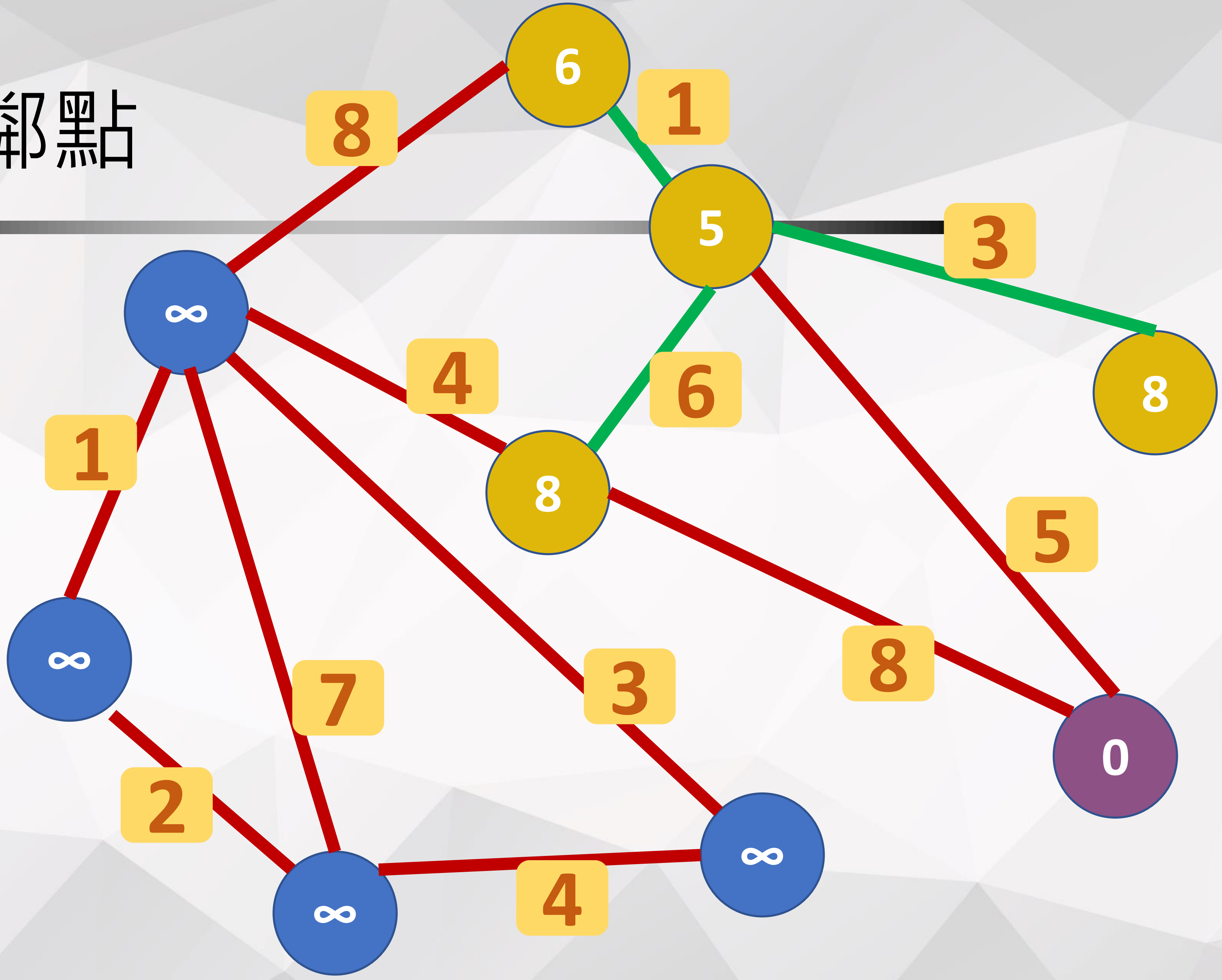

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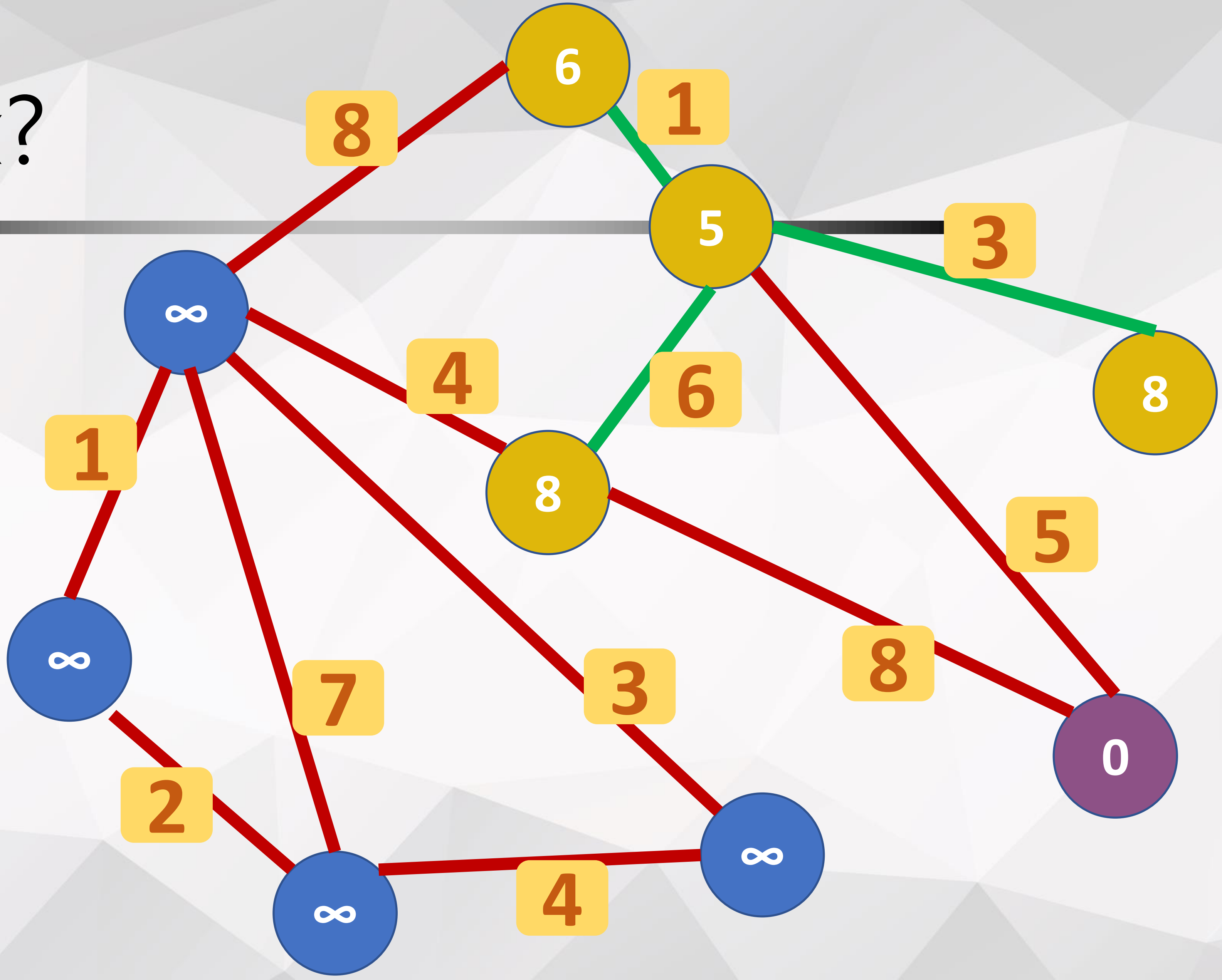

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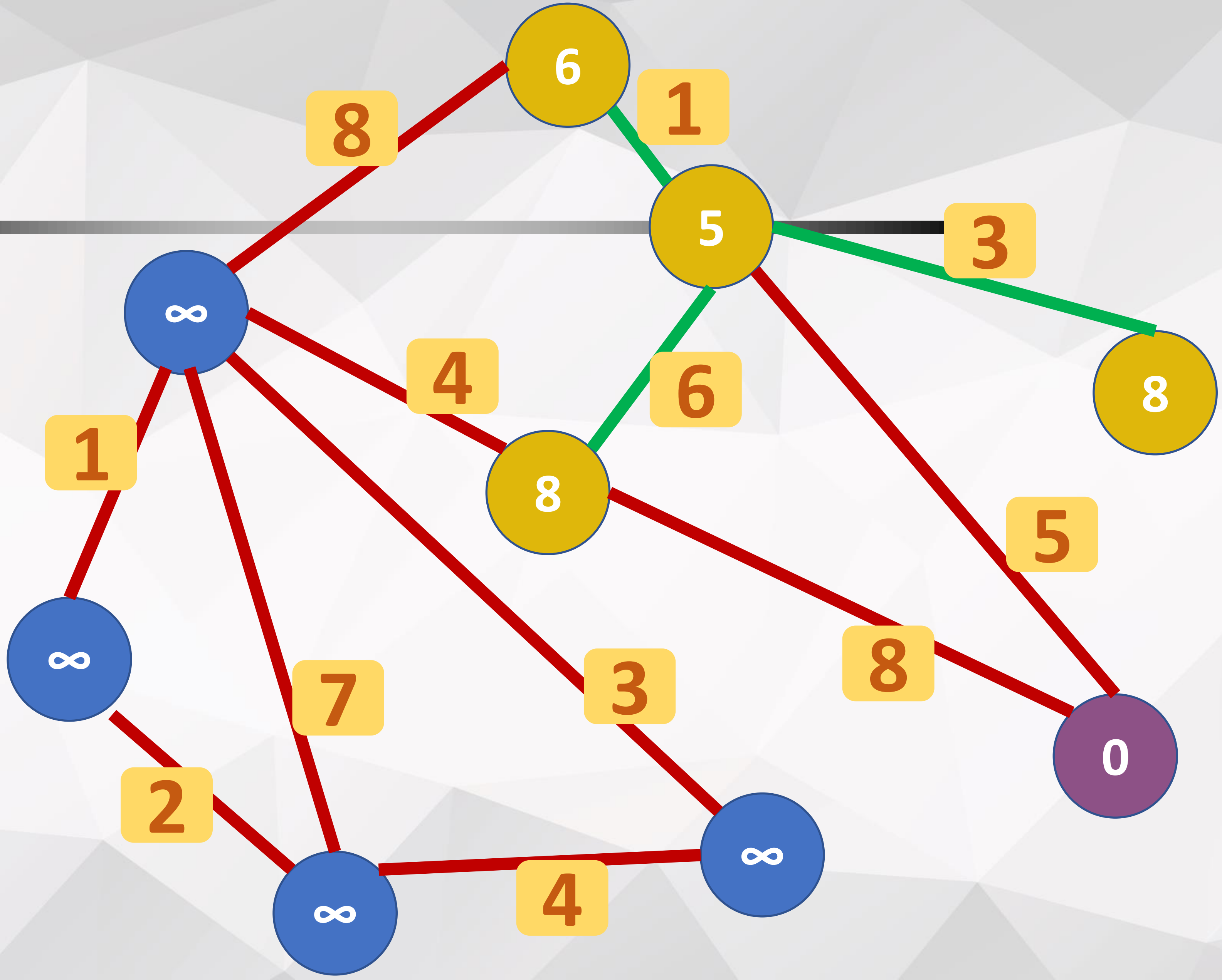
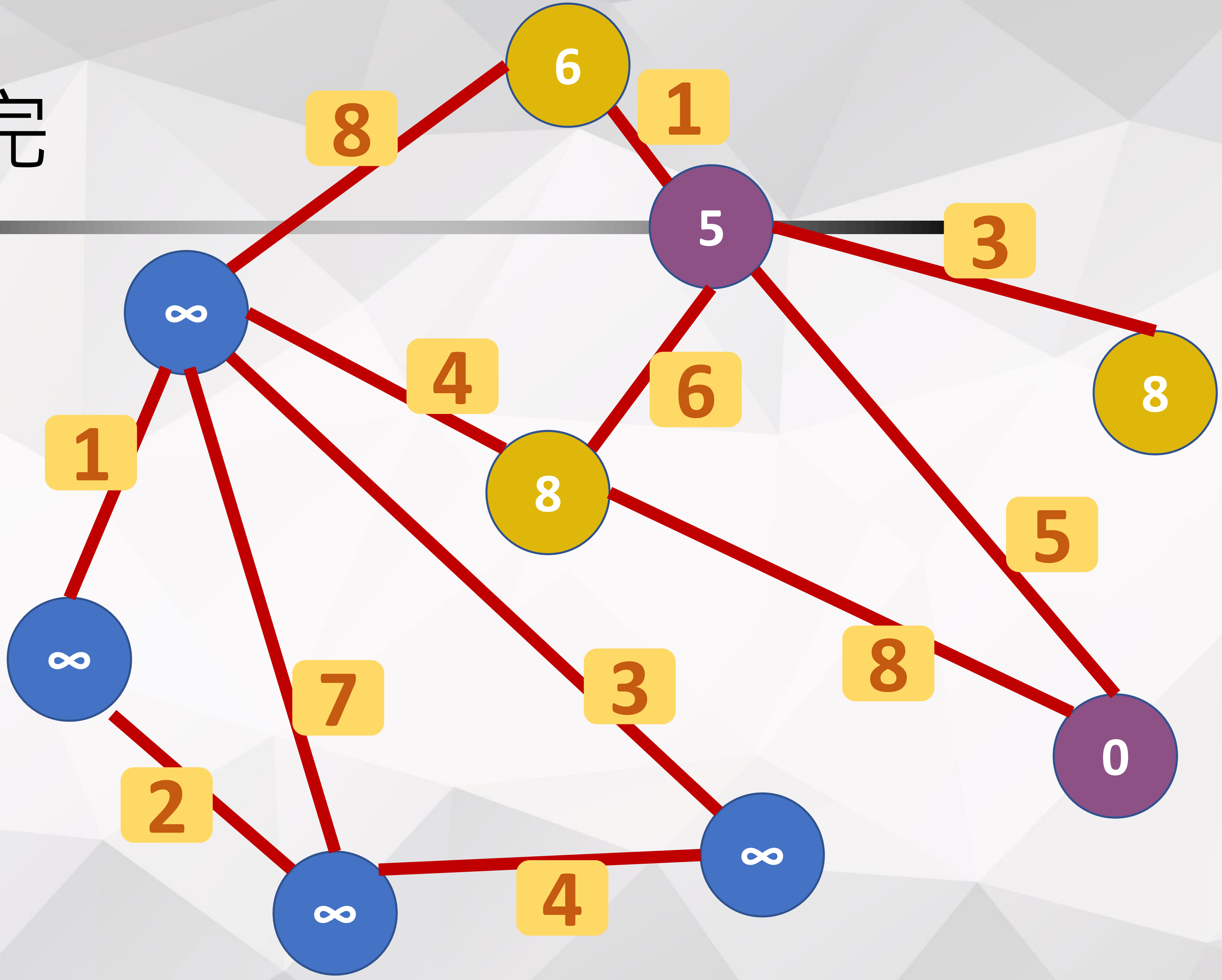


拜訪完



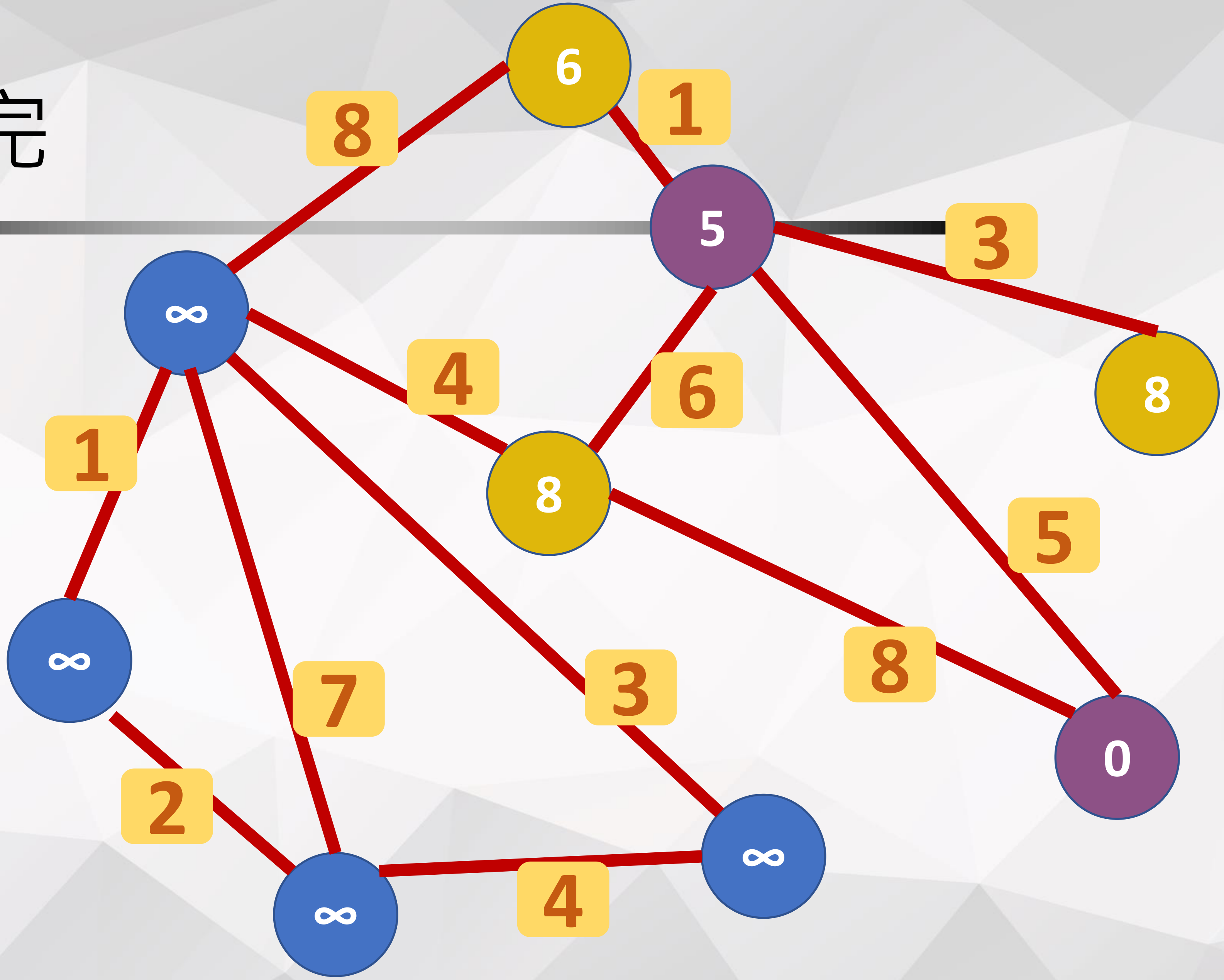
關於 relaxation

5

這個**拜訪完**的節點

在**未來**有可能再被 relax 嗎?

拜訪完



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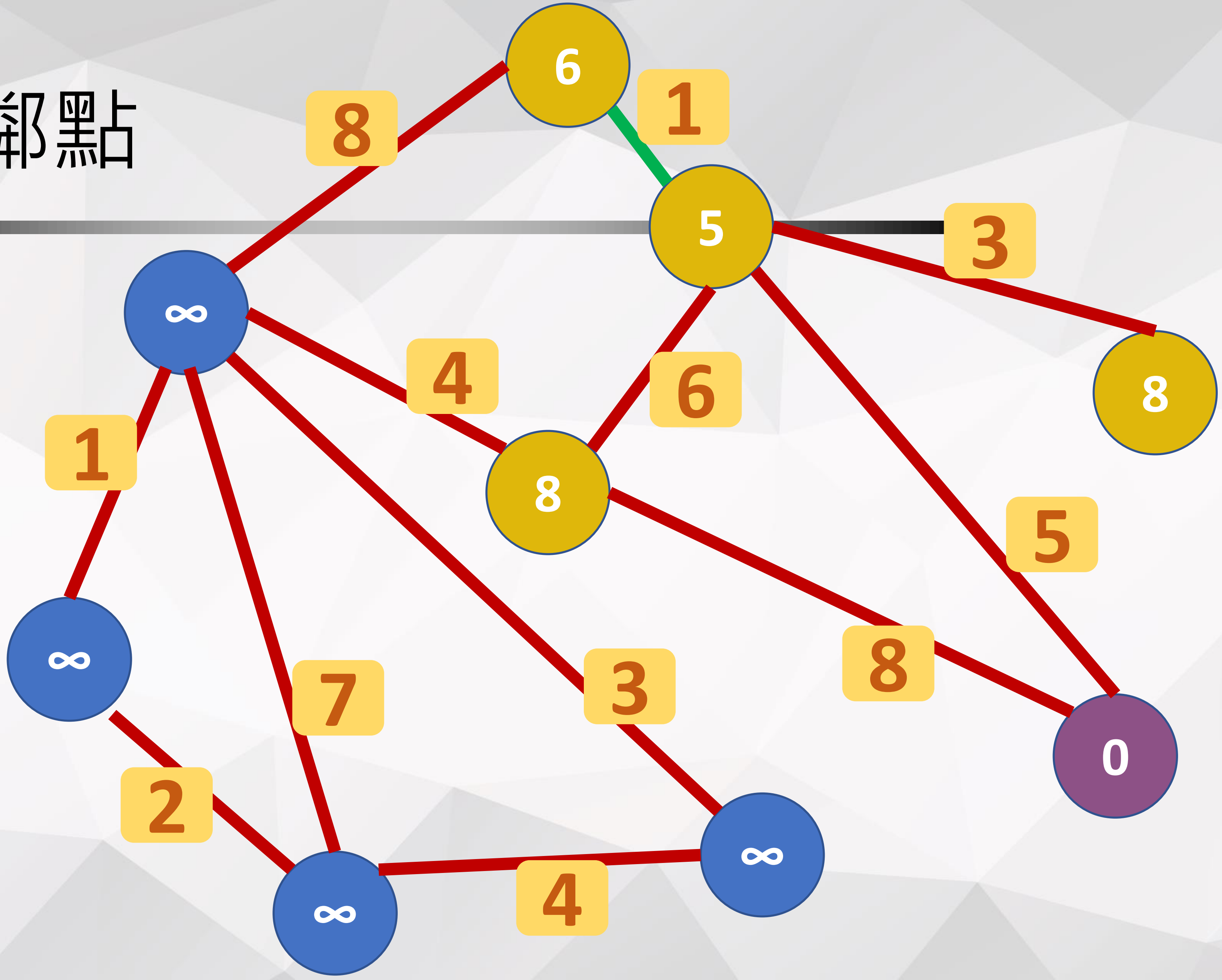

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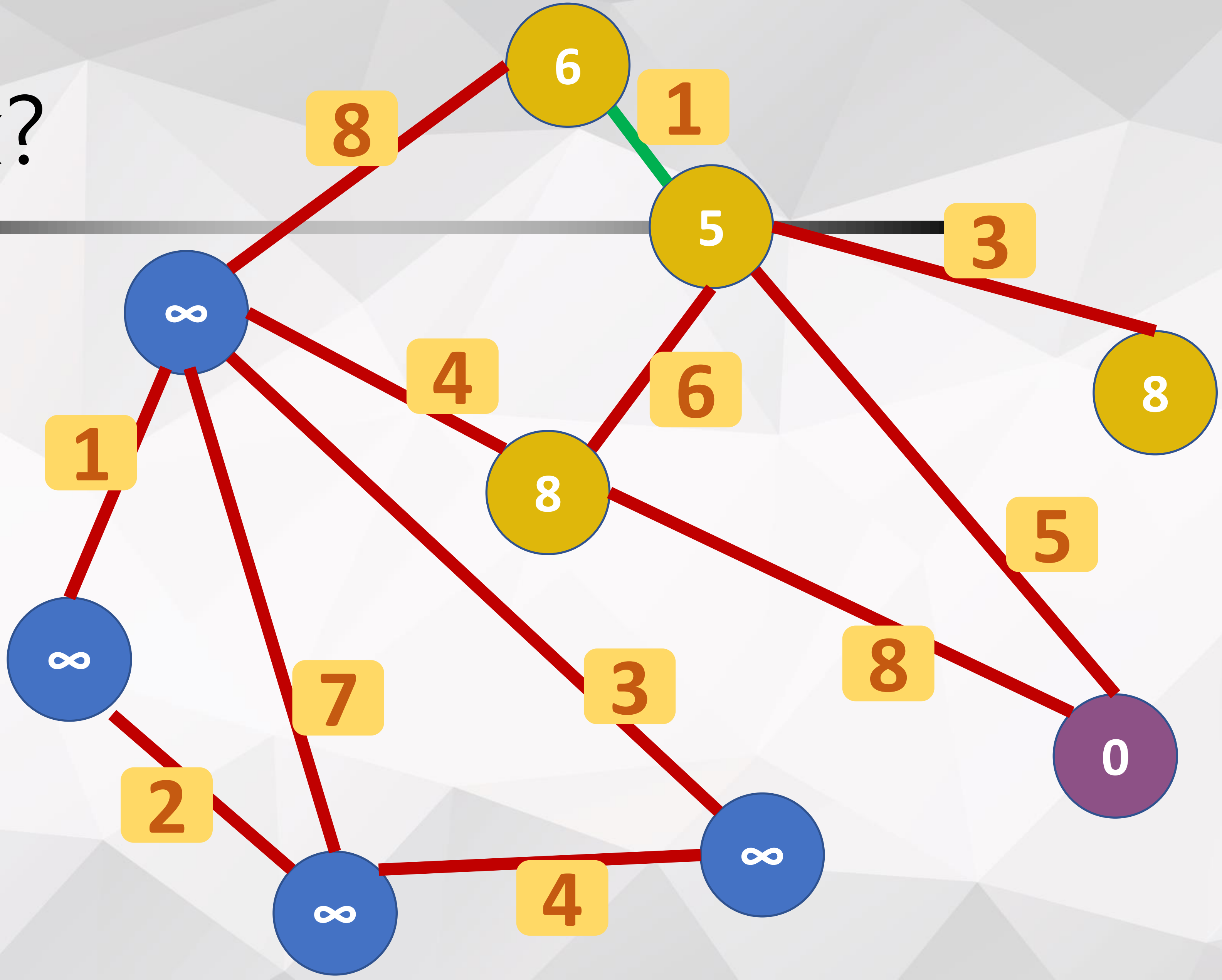

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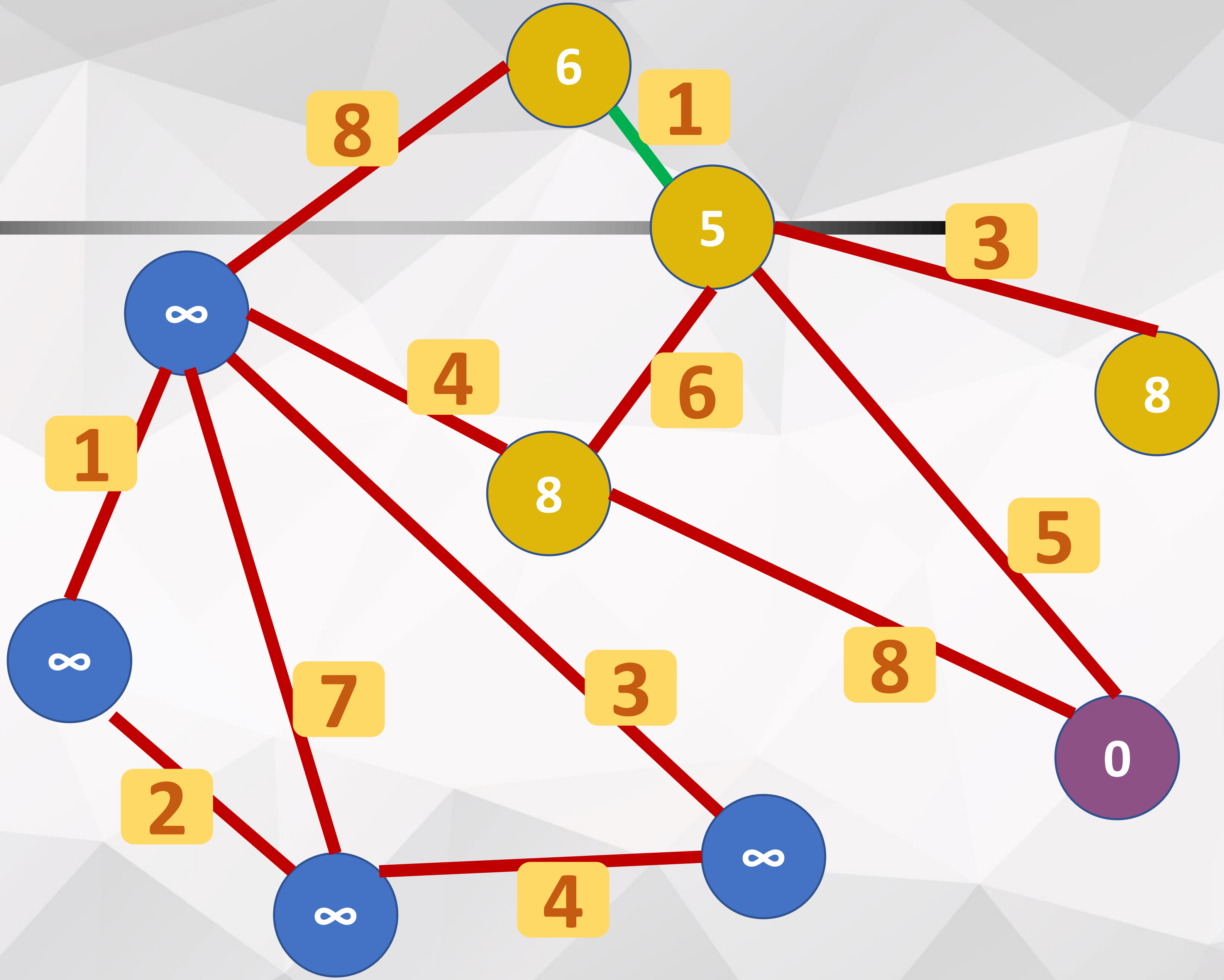

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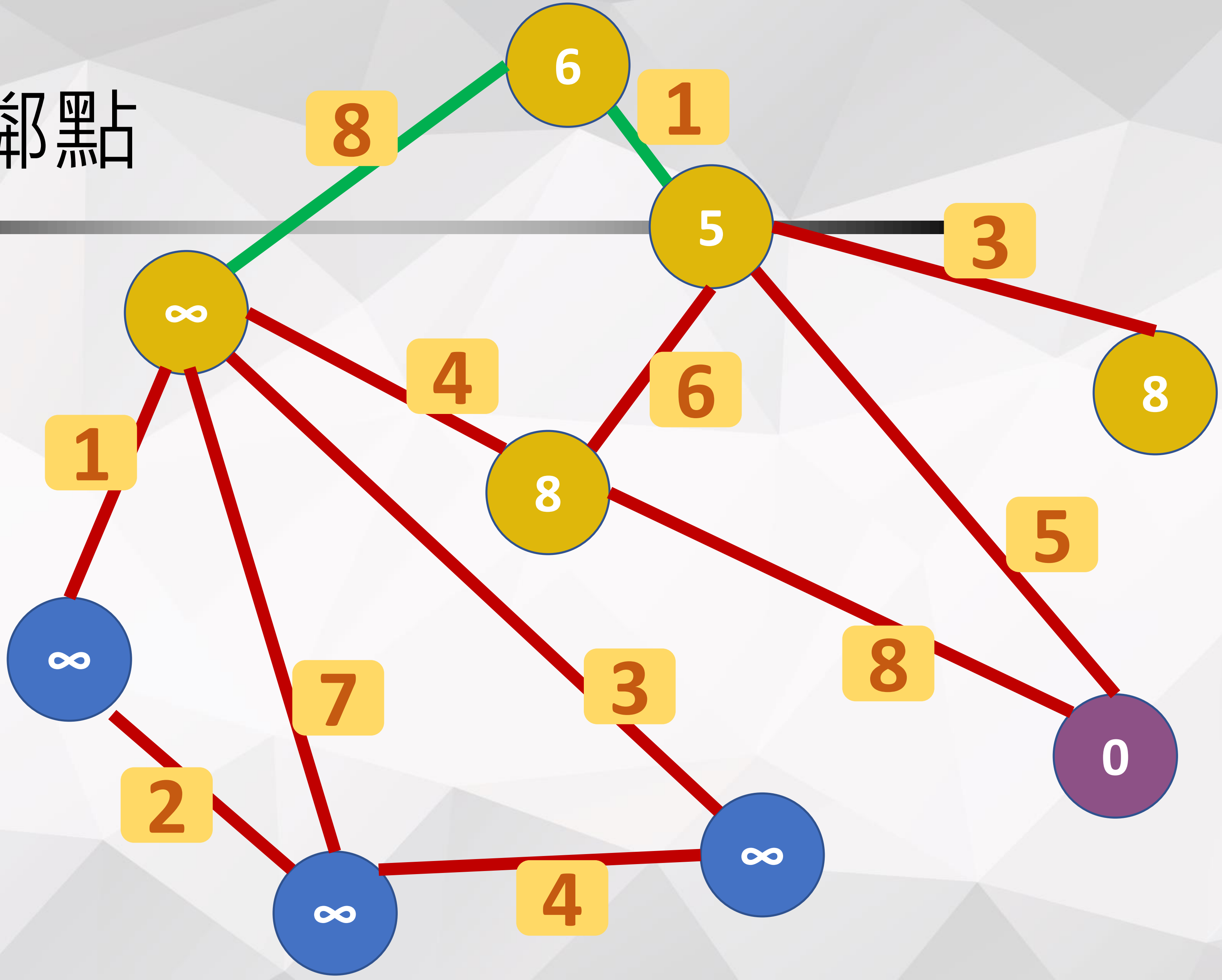

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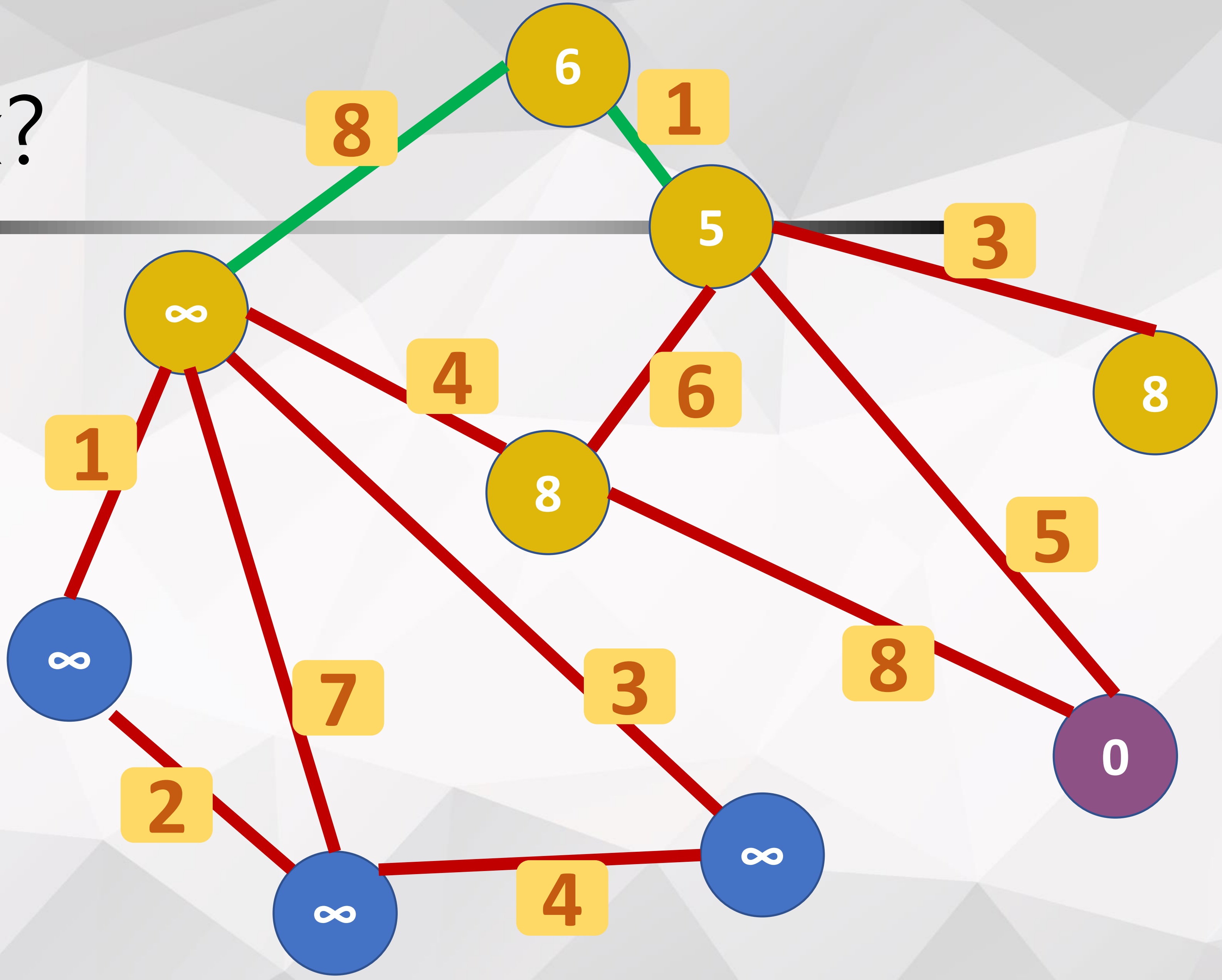

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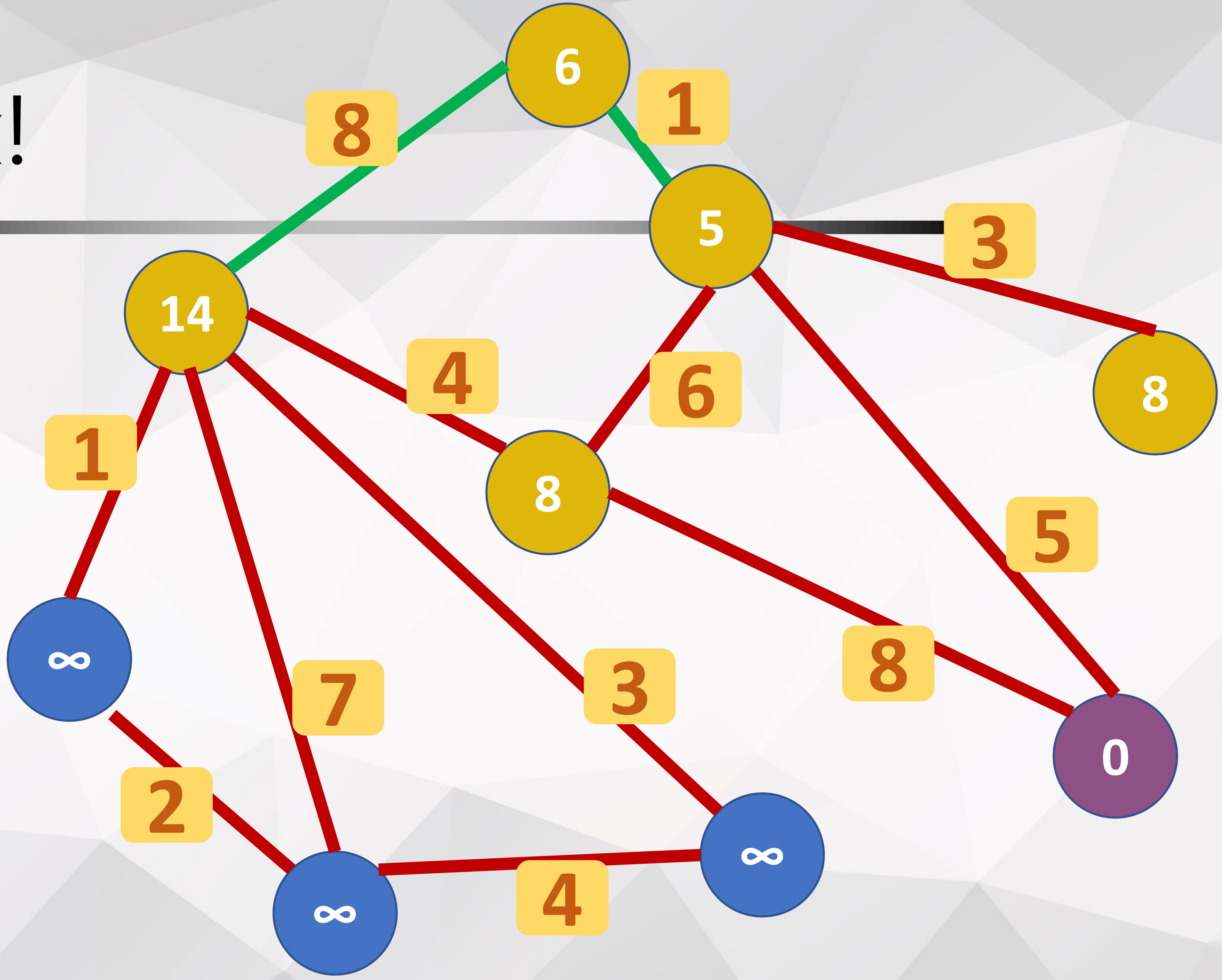

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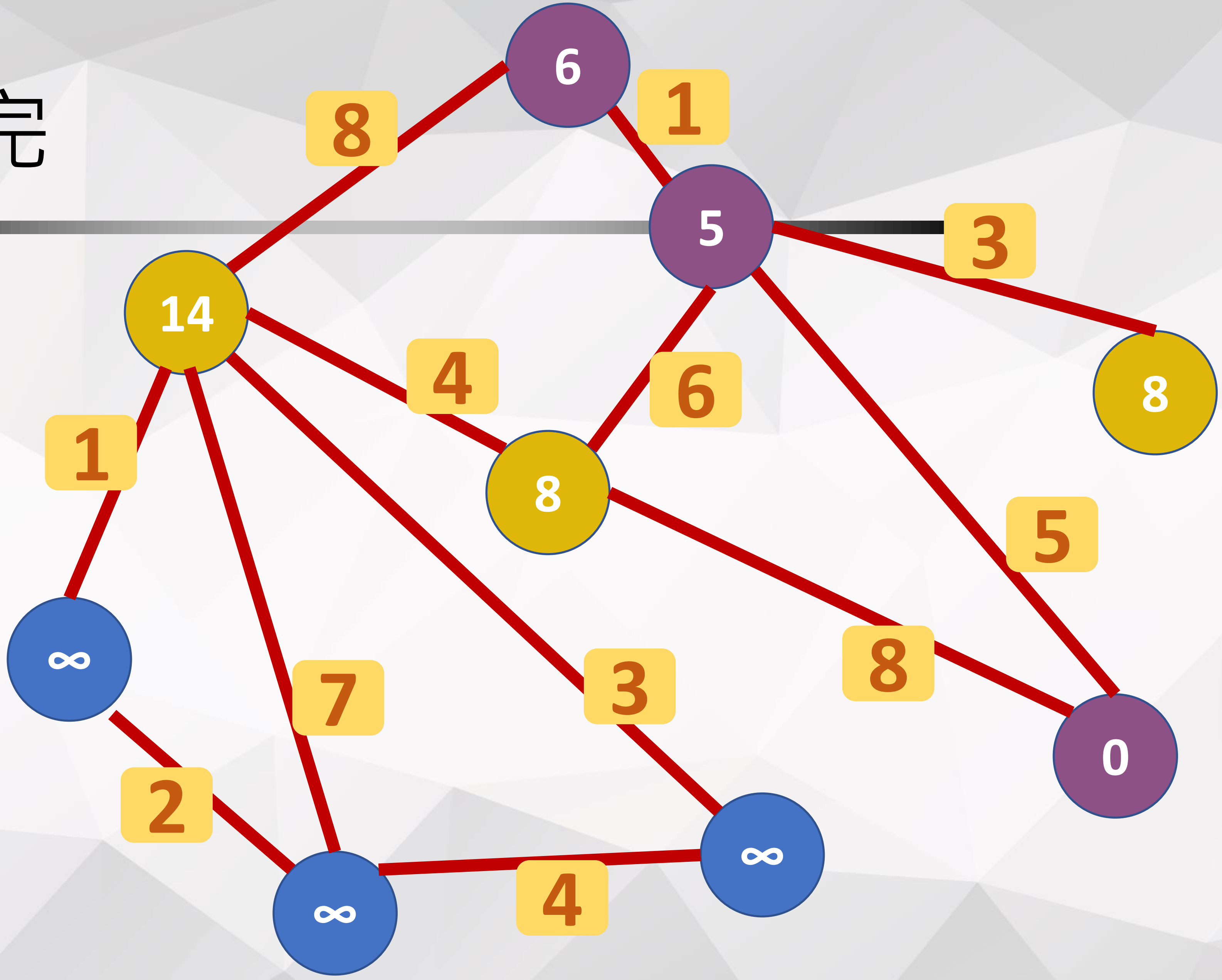


關於 relaxation

6

5

這些**拜訪完**的節點

在**未來**有可能再被 relax 嗎?

0

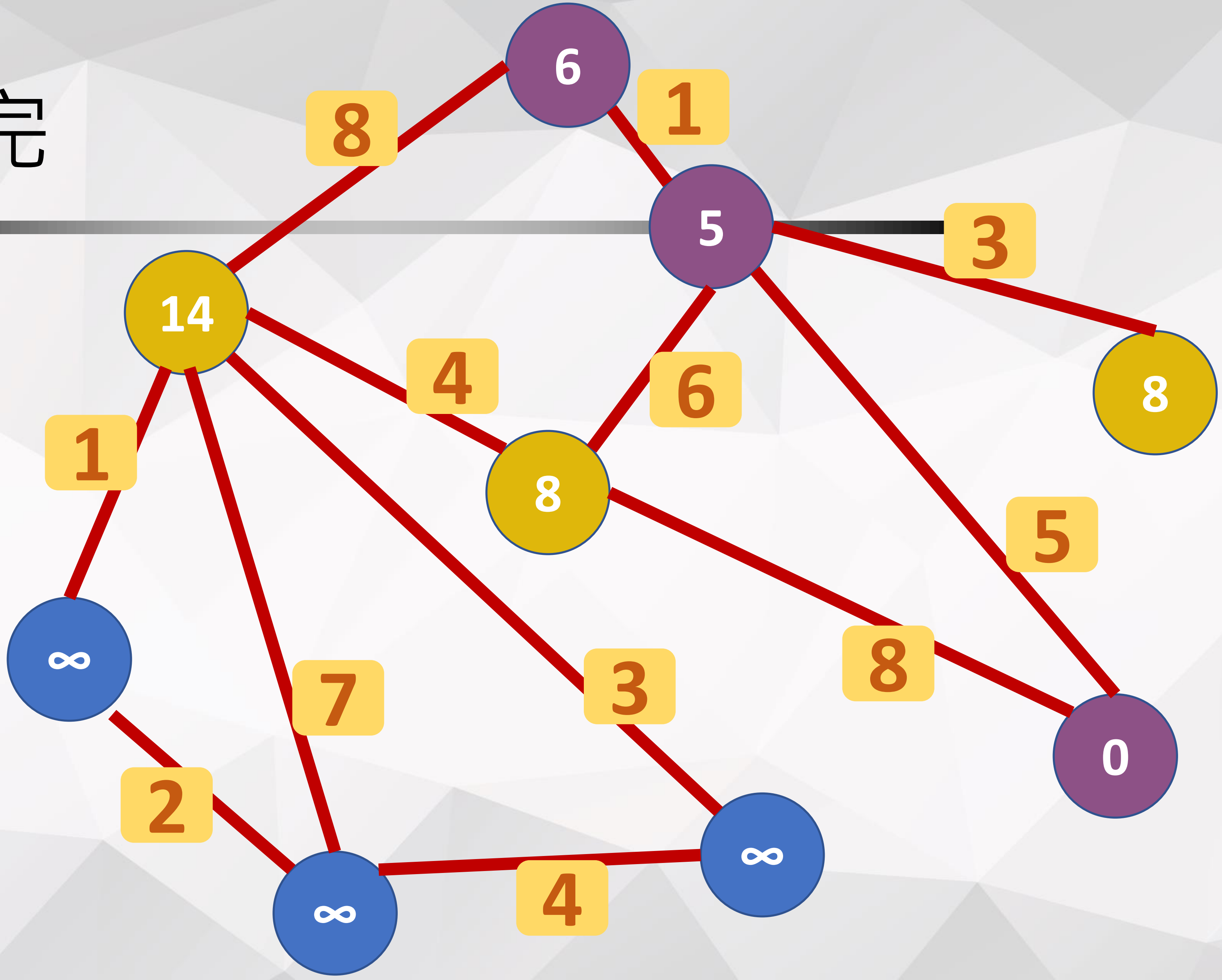
關於 relaxation

這些**拜訪完**的節點

在**未來**有可能再被 relax 嗎?

若答案為**否定**的，
這似乎叫做: **無後效性**

拜訪完



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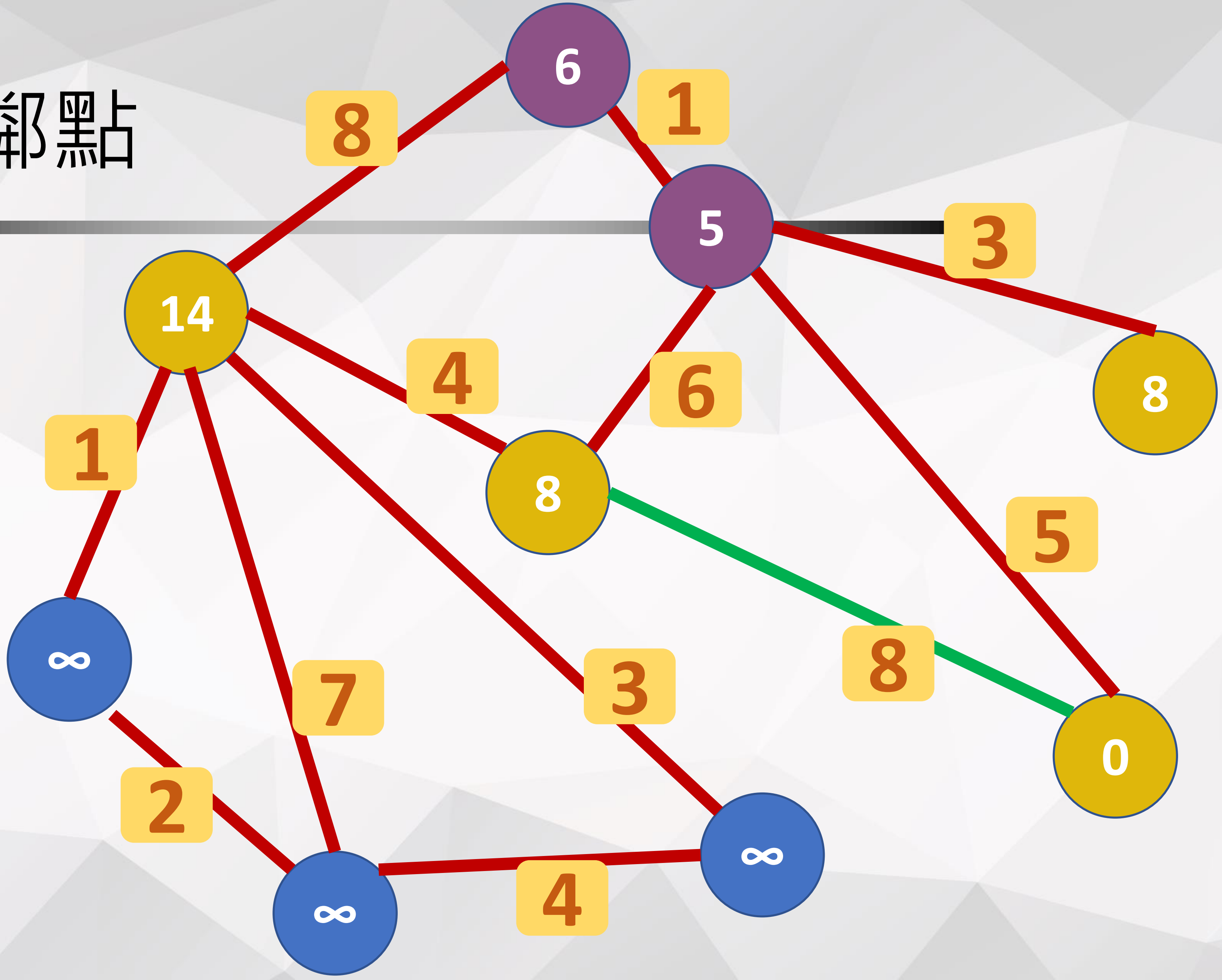

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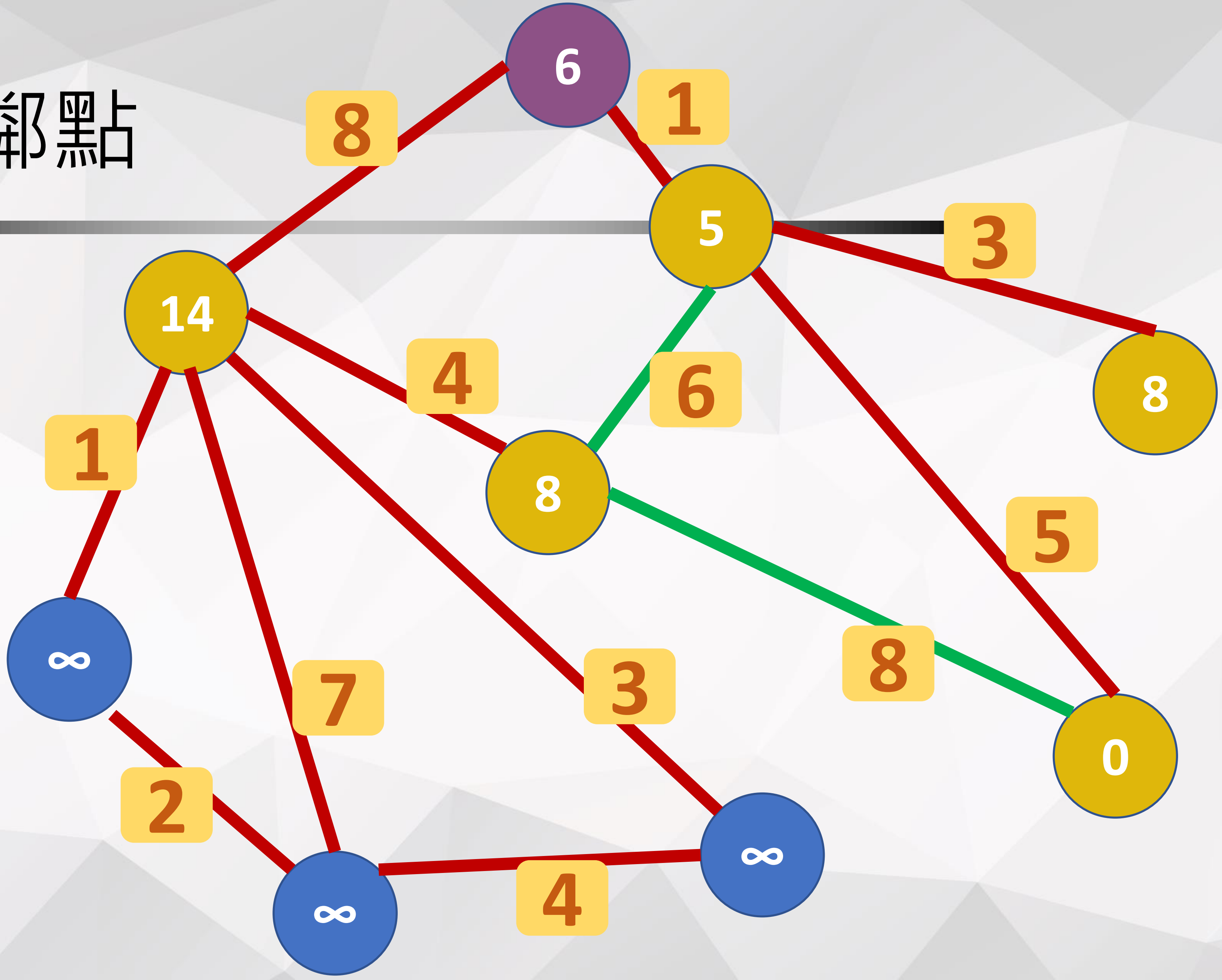

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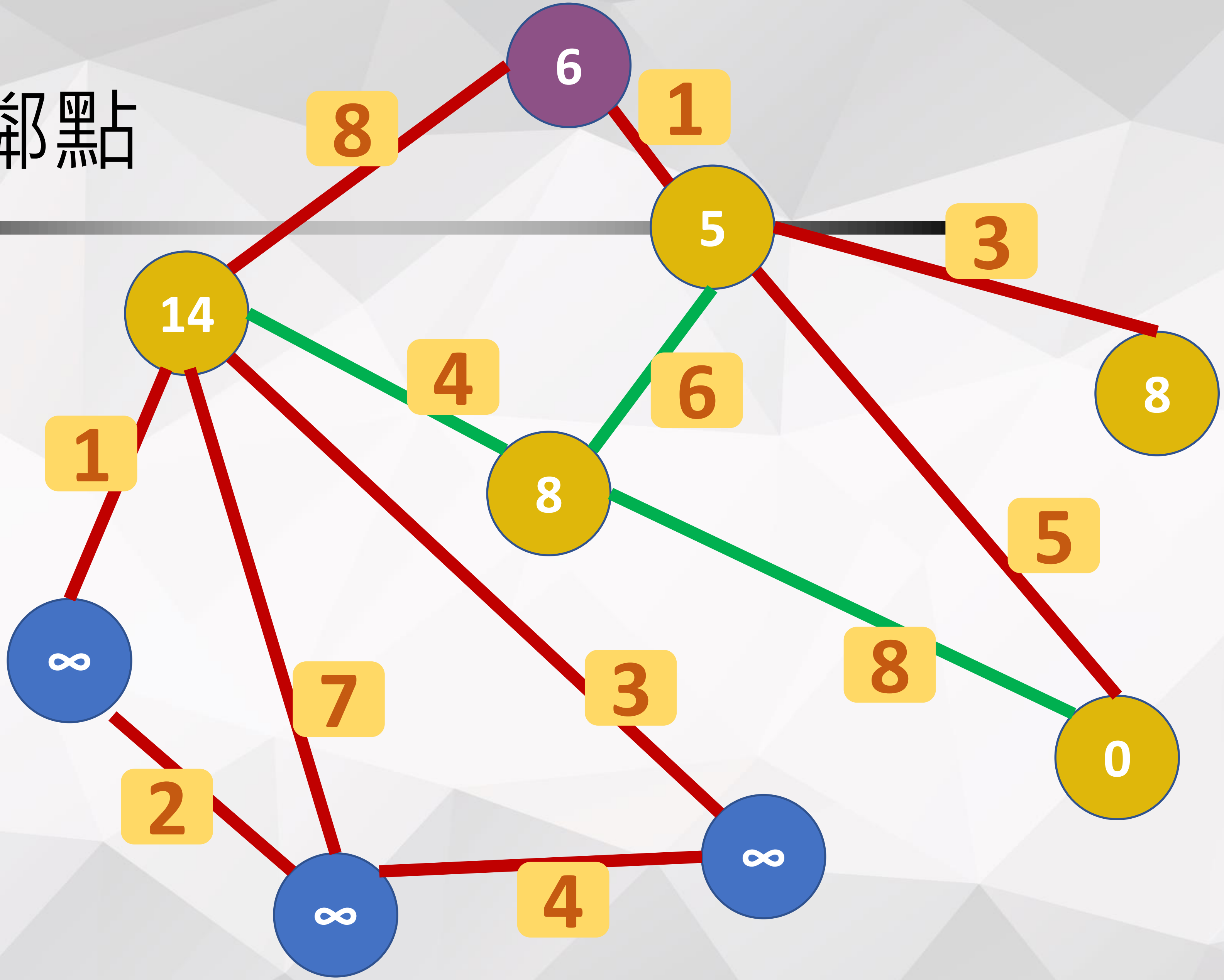

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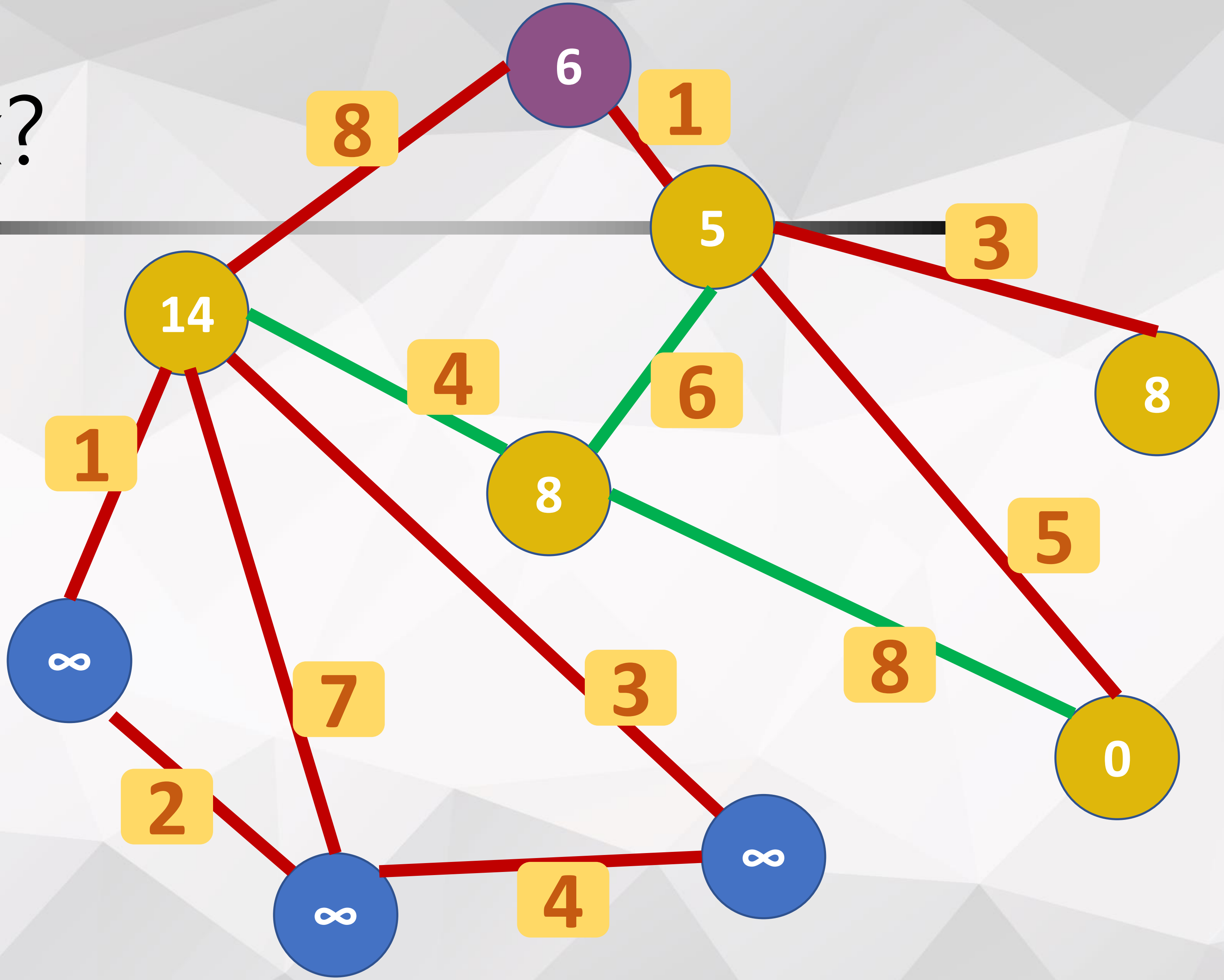

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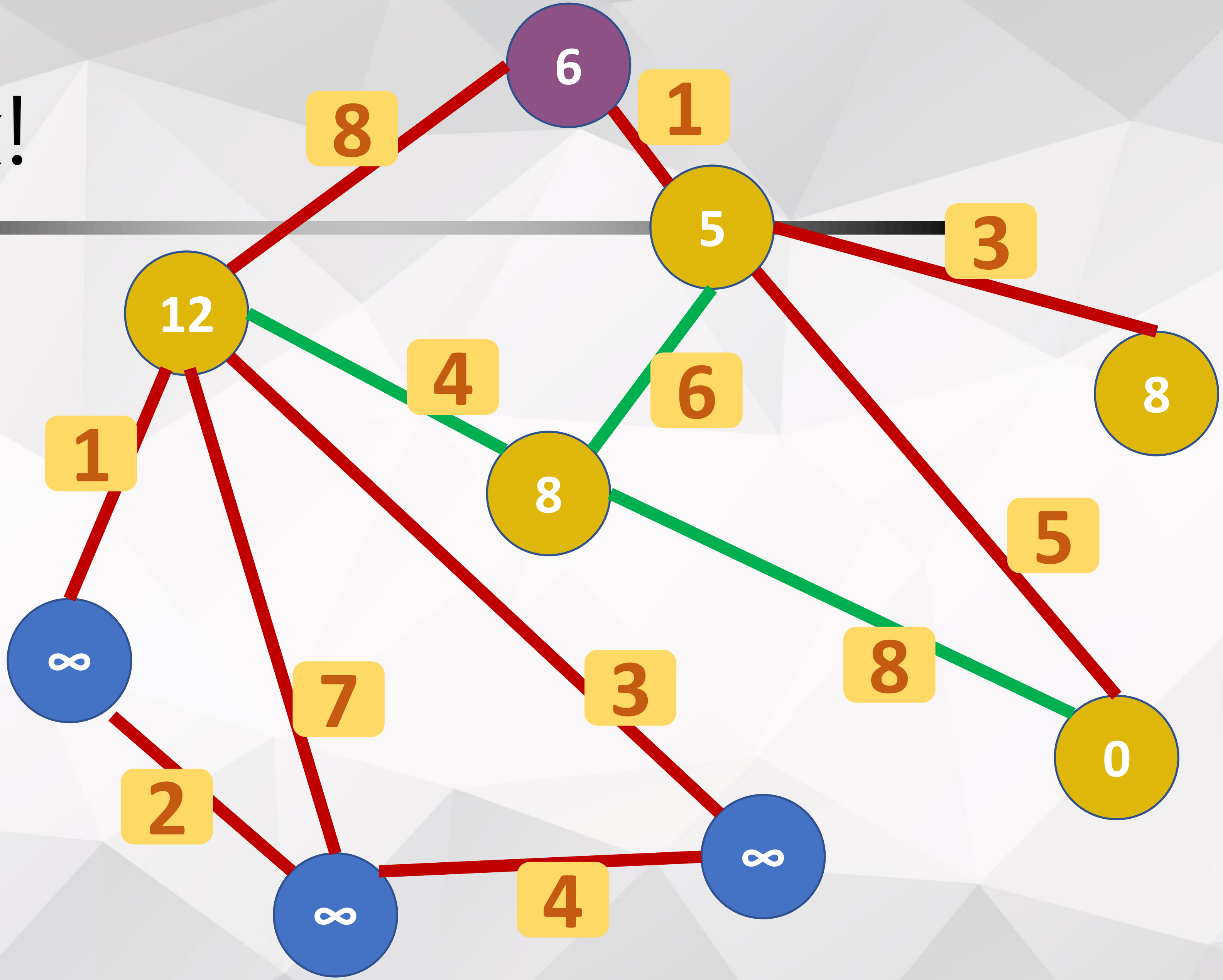

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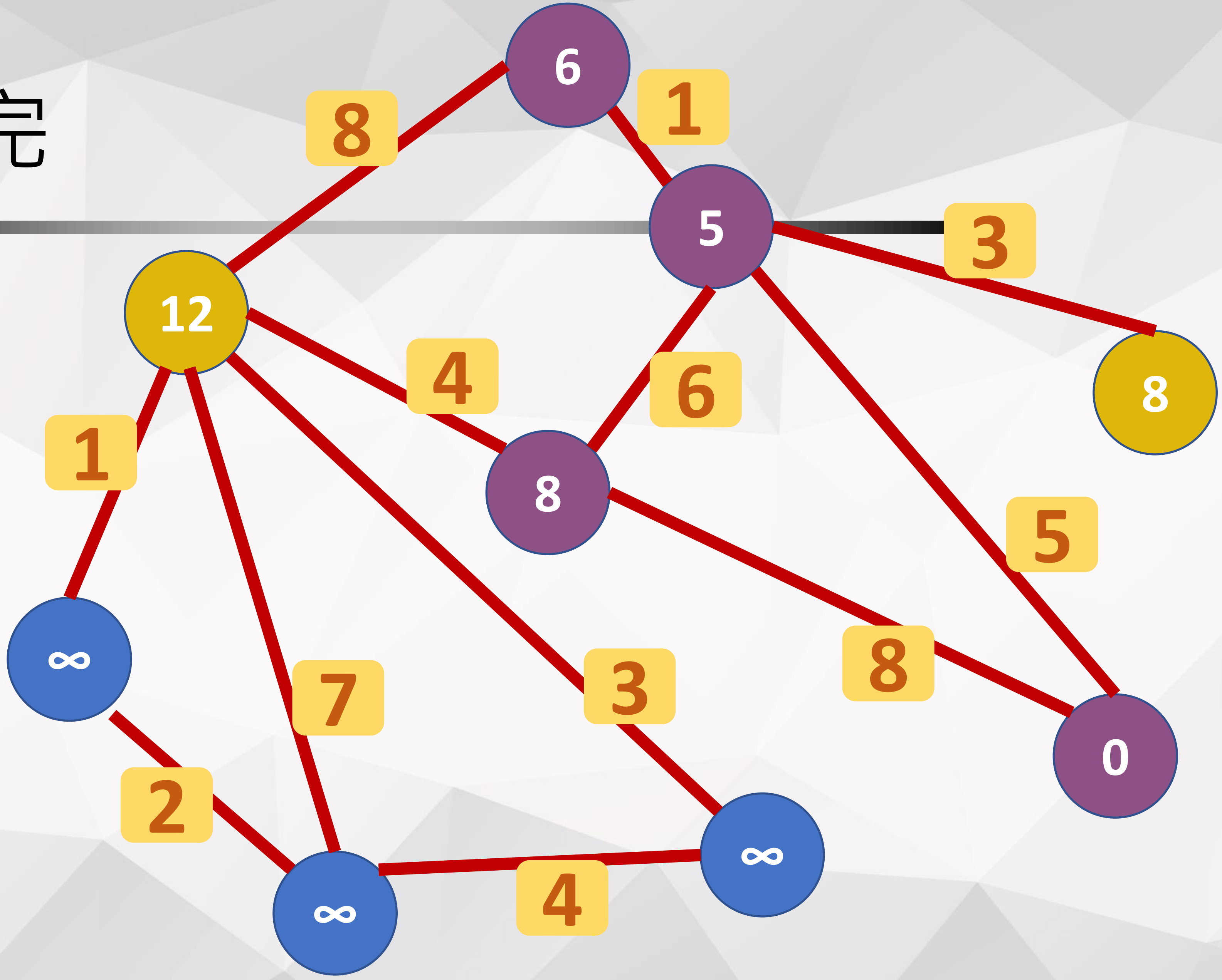

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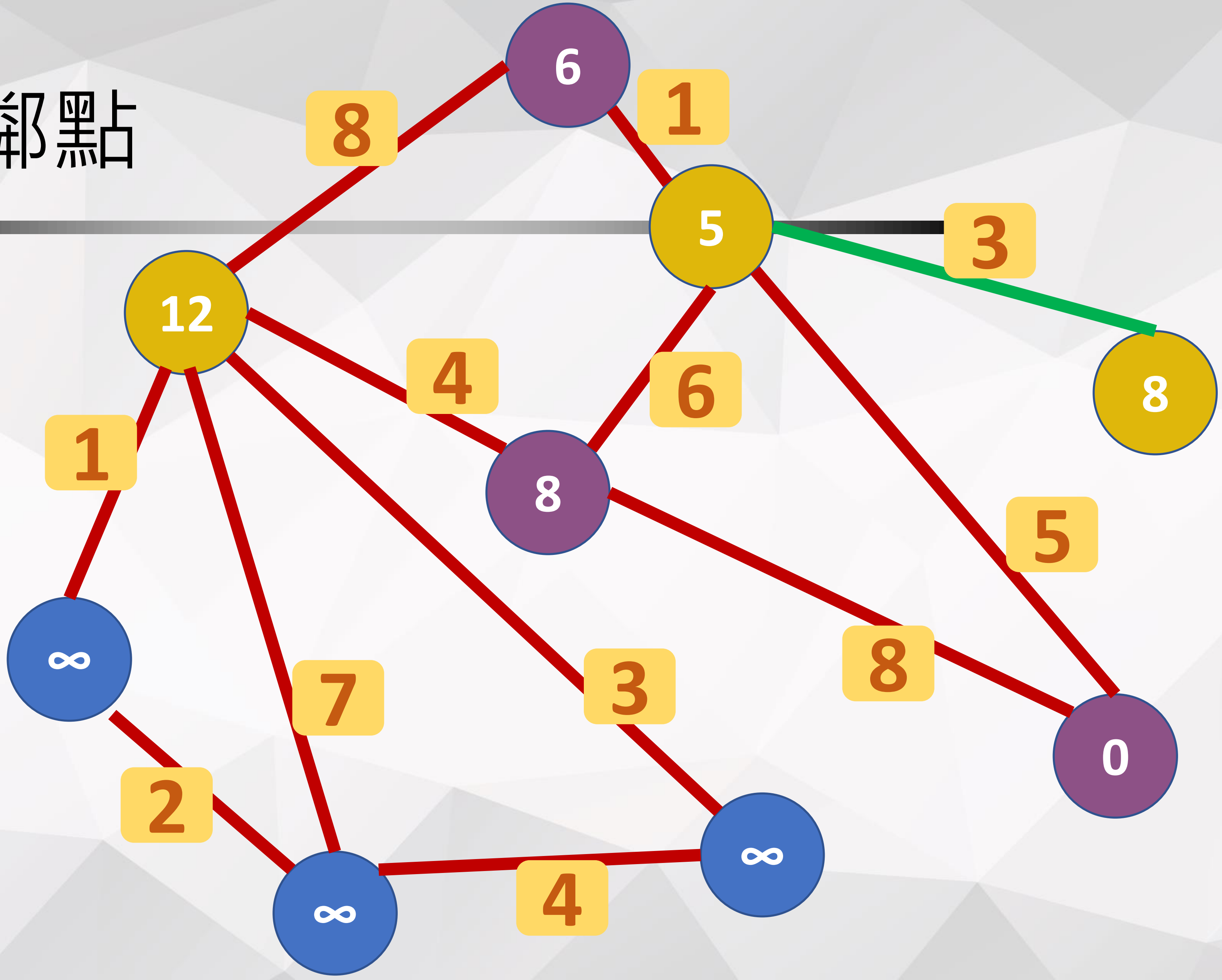

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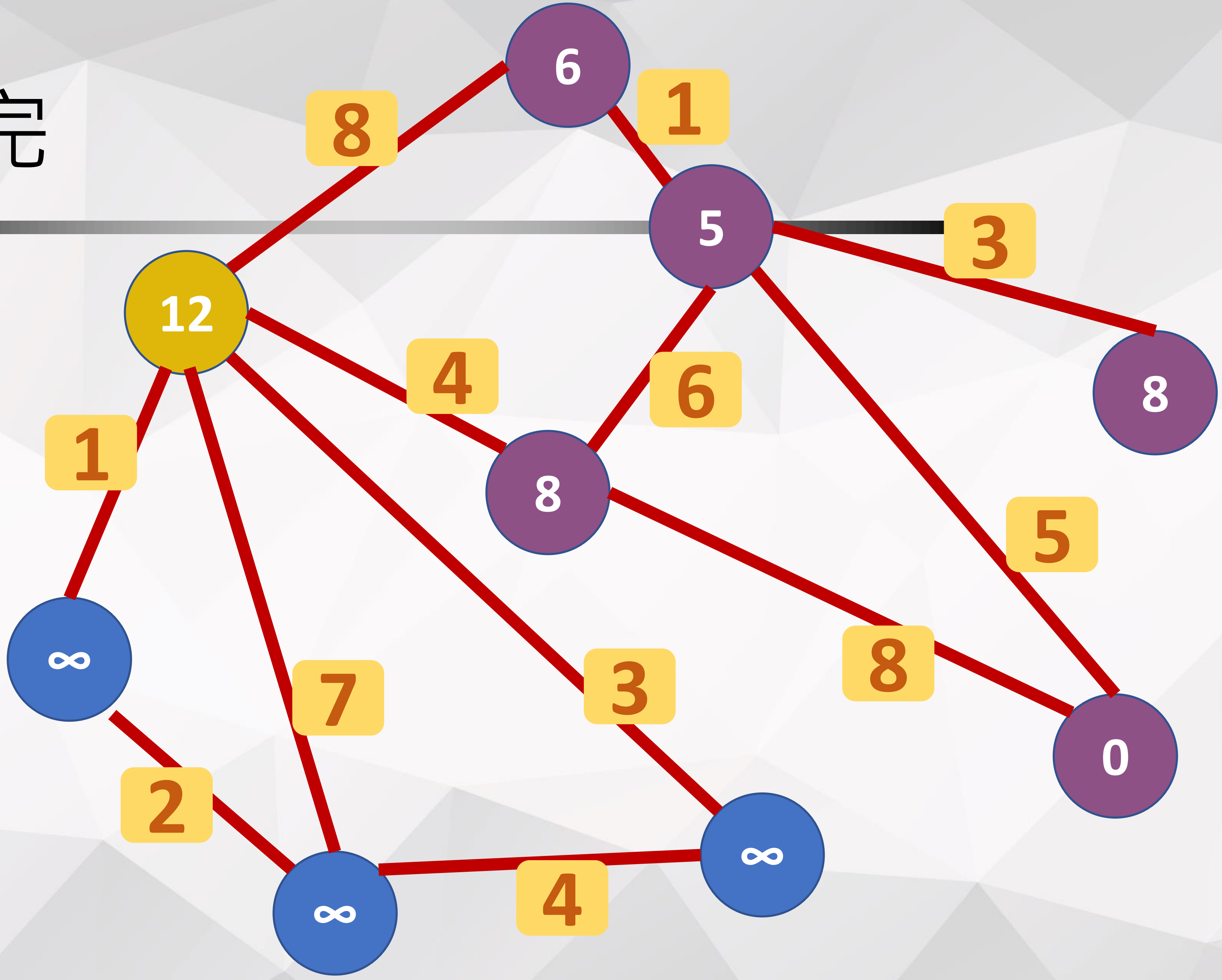

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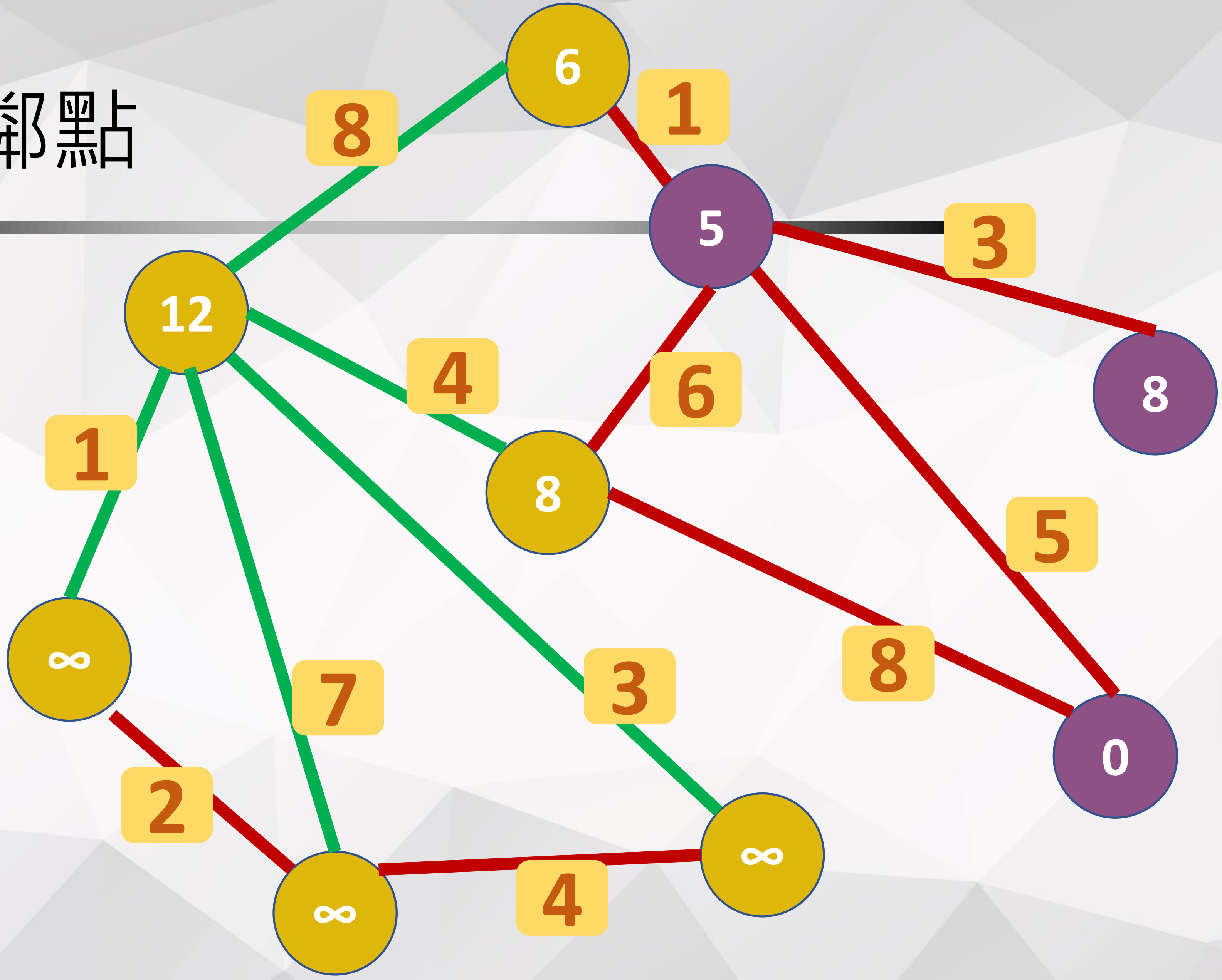

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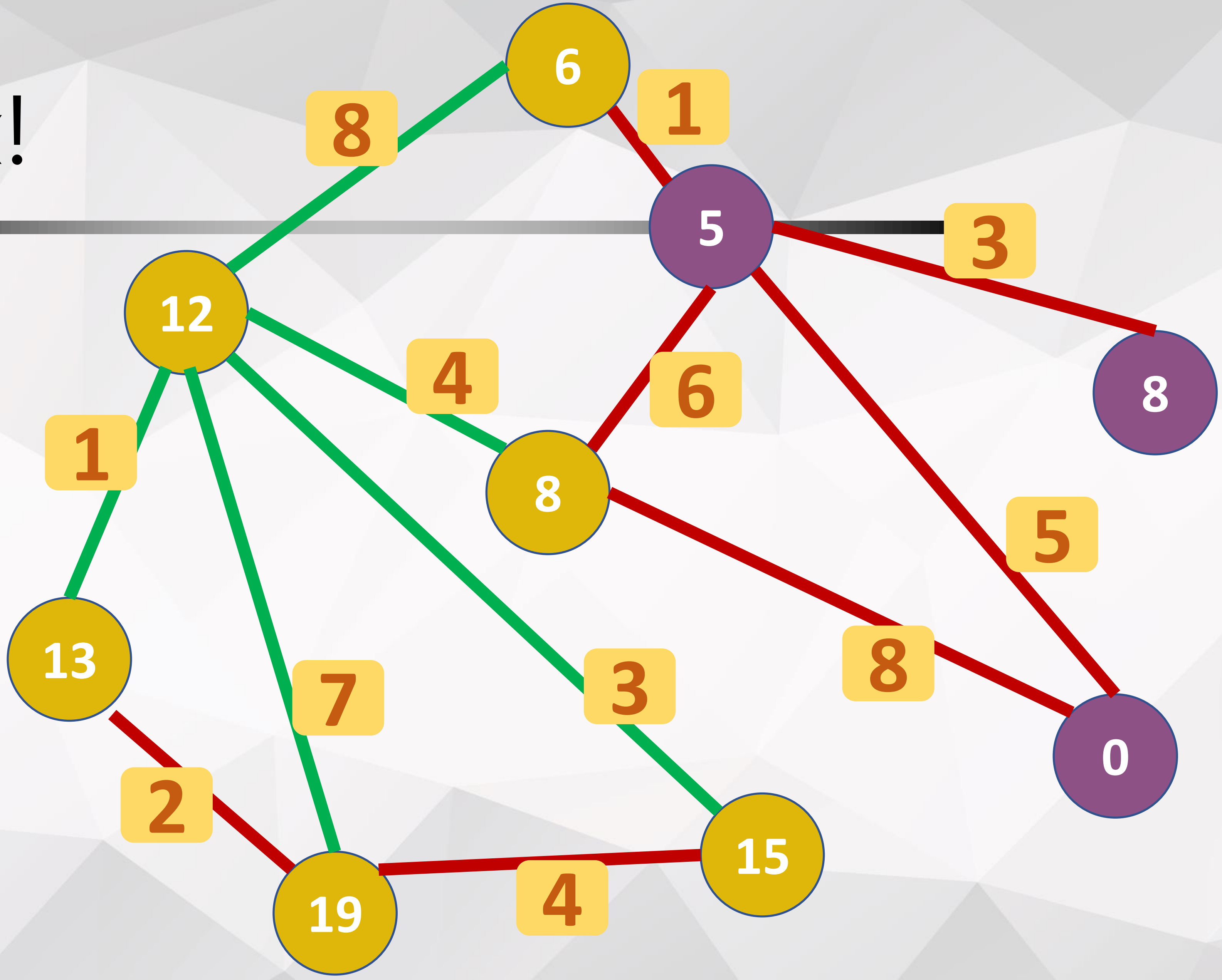

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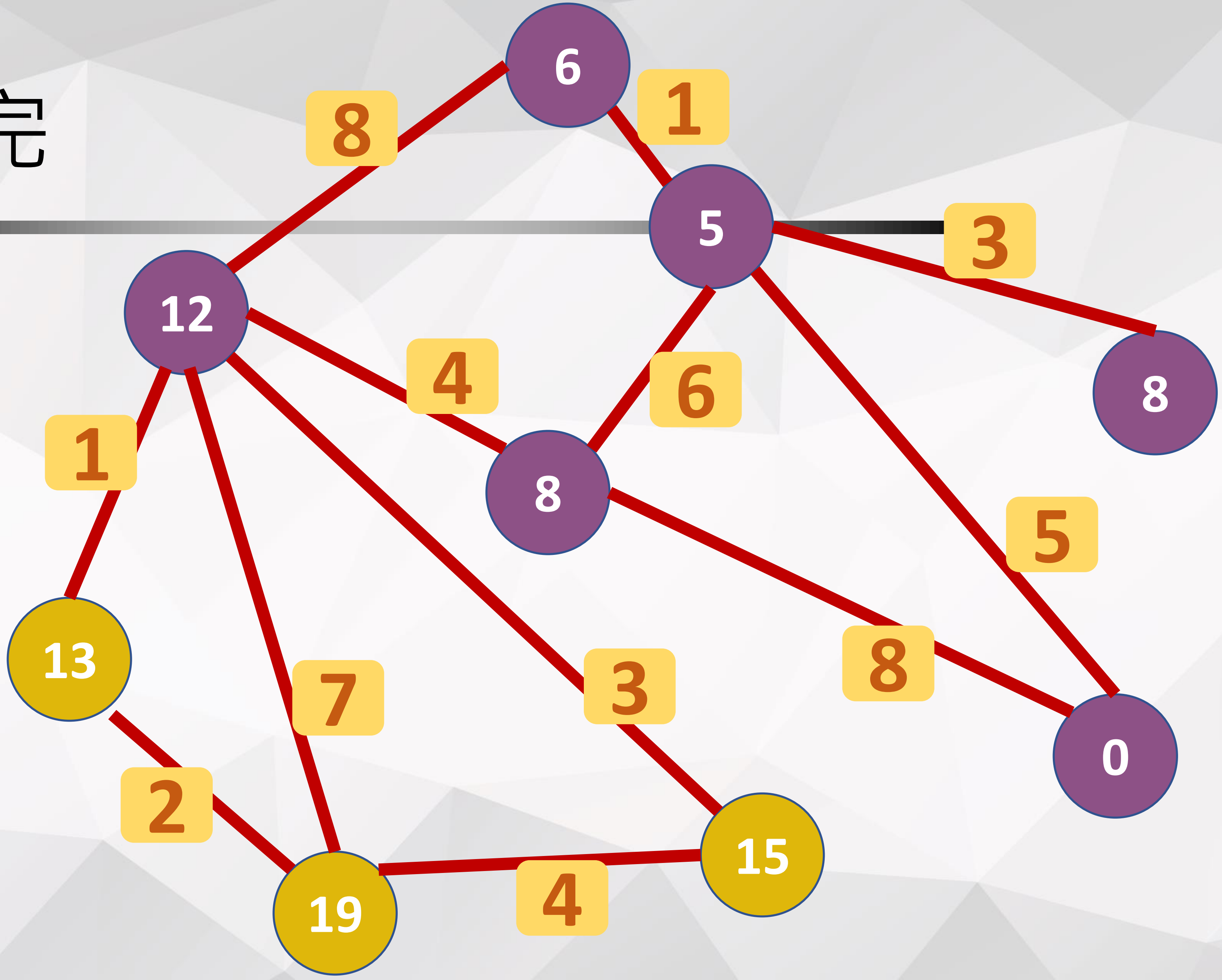
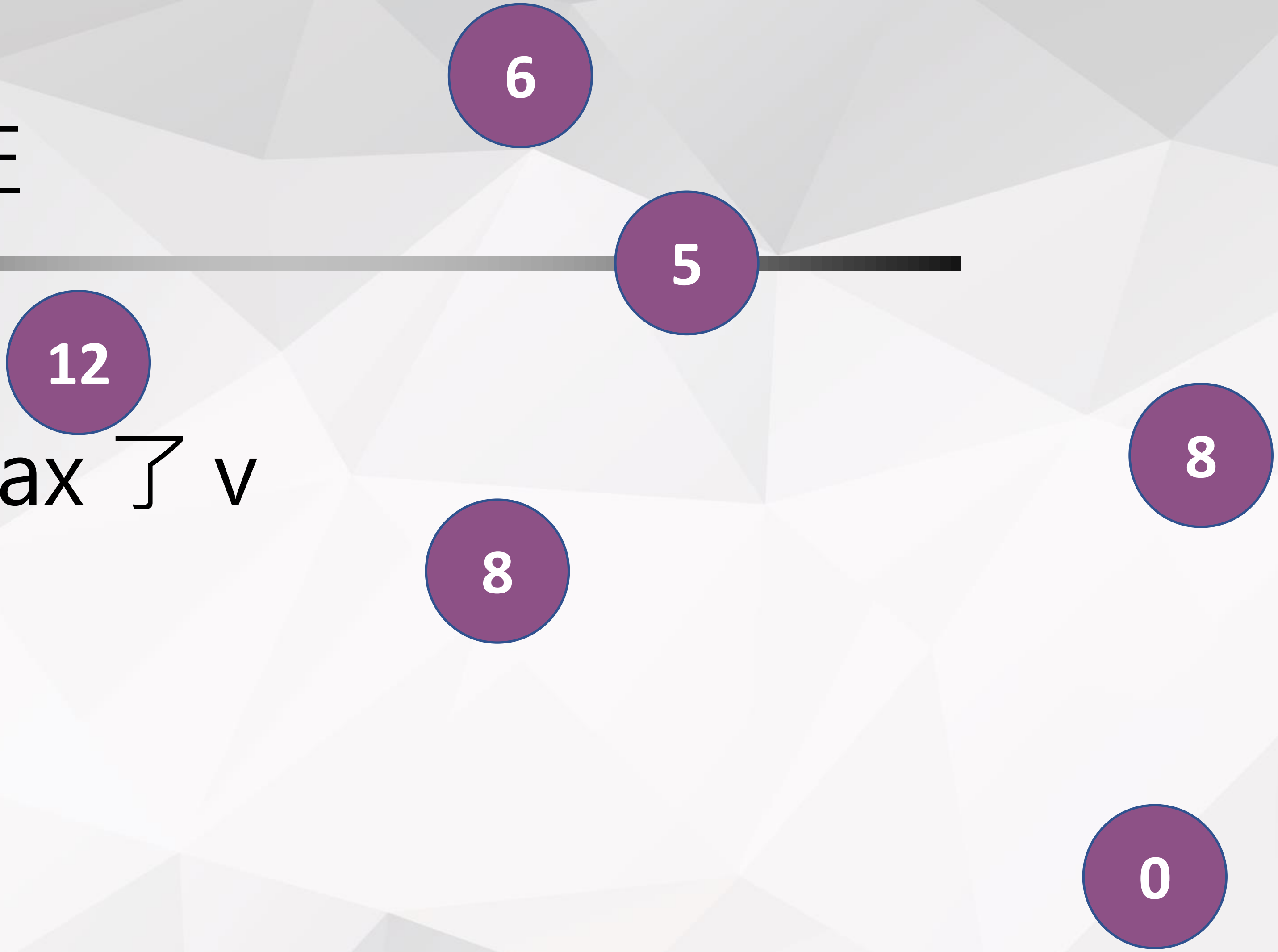


無後效性

若 u 去 relax 了 v



無後效性

若 u 去 relax 了 v

則未來不管如何，

v 不可能更新 u

無後效性

若 u 去 relax 了 v

則未來不管如何，

v 不可能更新 u

因為 u 先來的

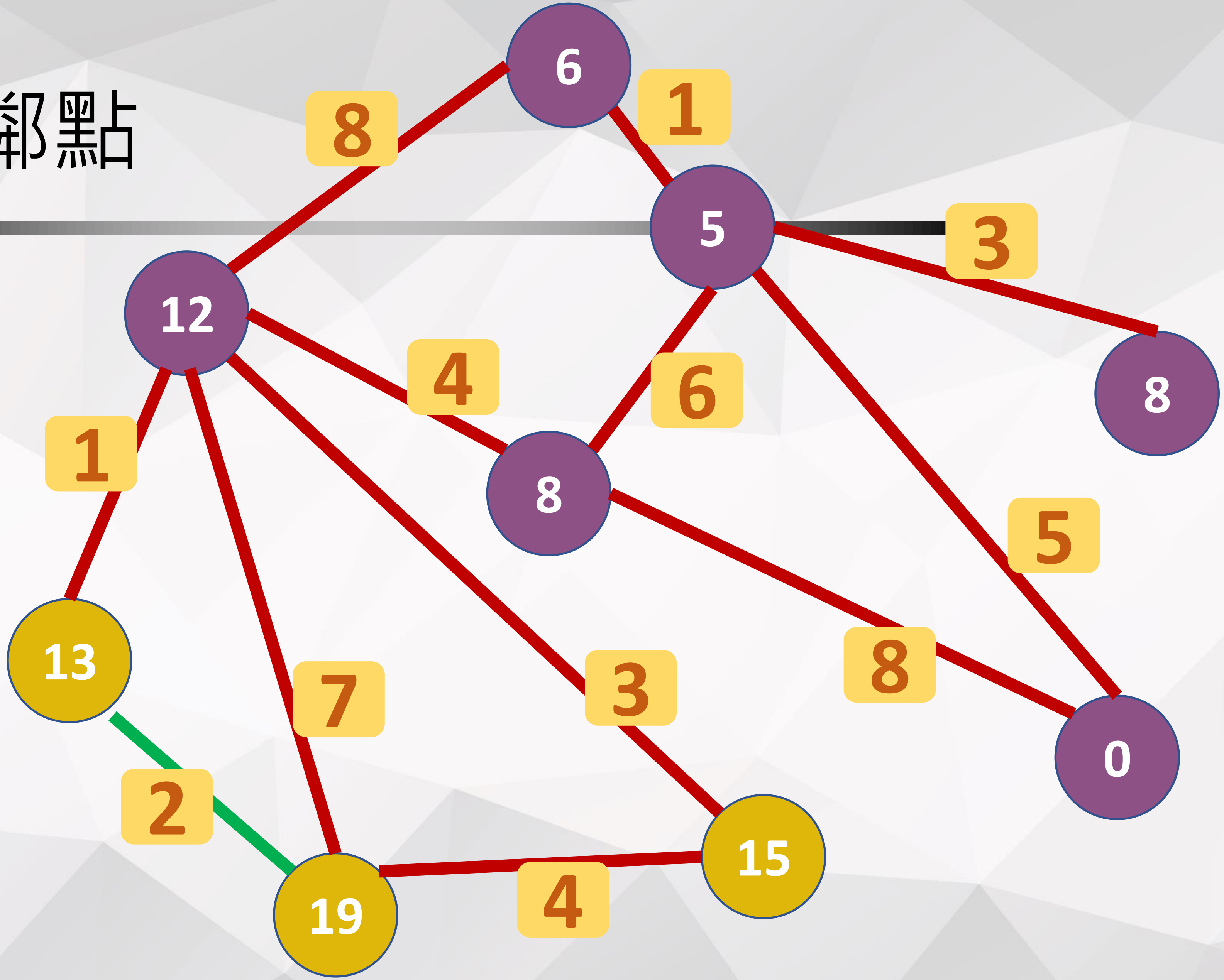
無後效性

若 u 去 relax 了 v
則未來不管如何，

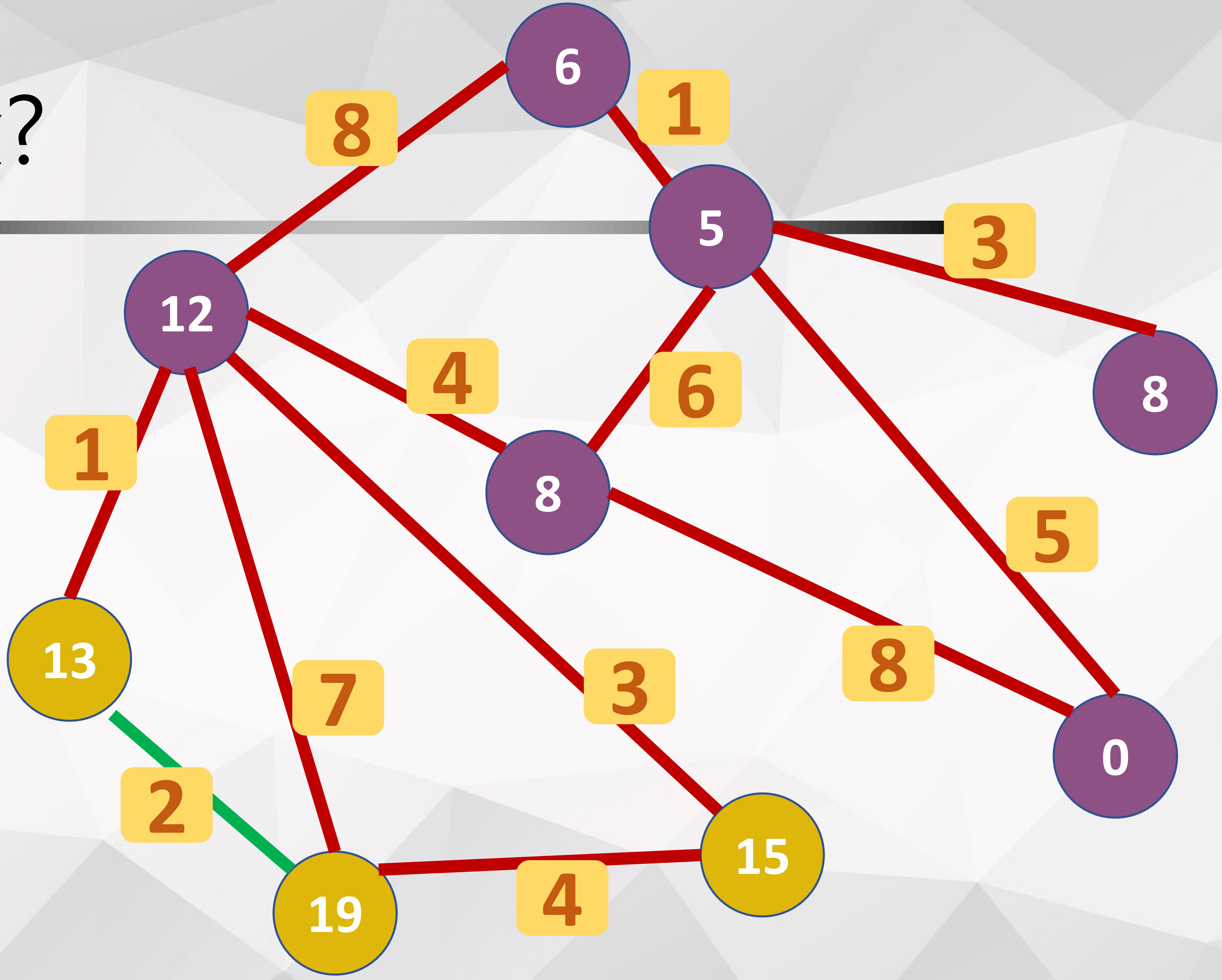
v 不可能更新 u

因為 u **先來的**，在 u 之後挑的**任何點**
其值都比 u 大，並且**邊權重恆正**！
怎麼**加**都比 u 大

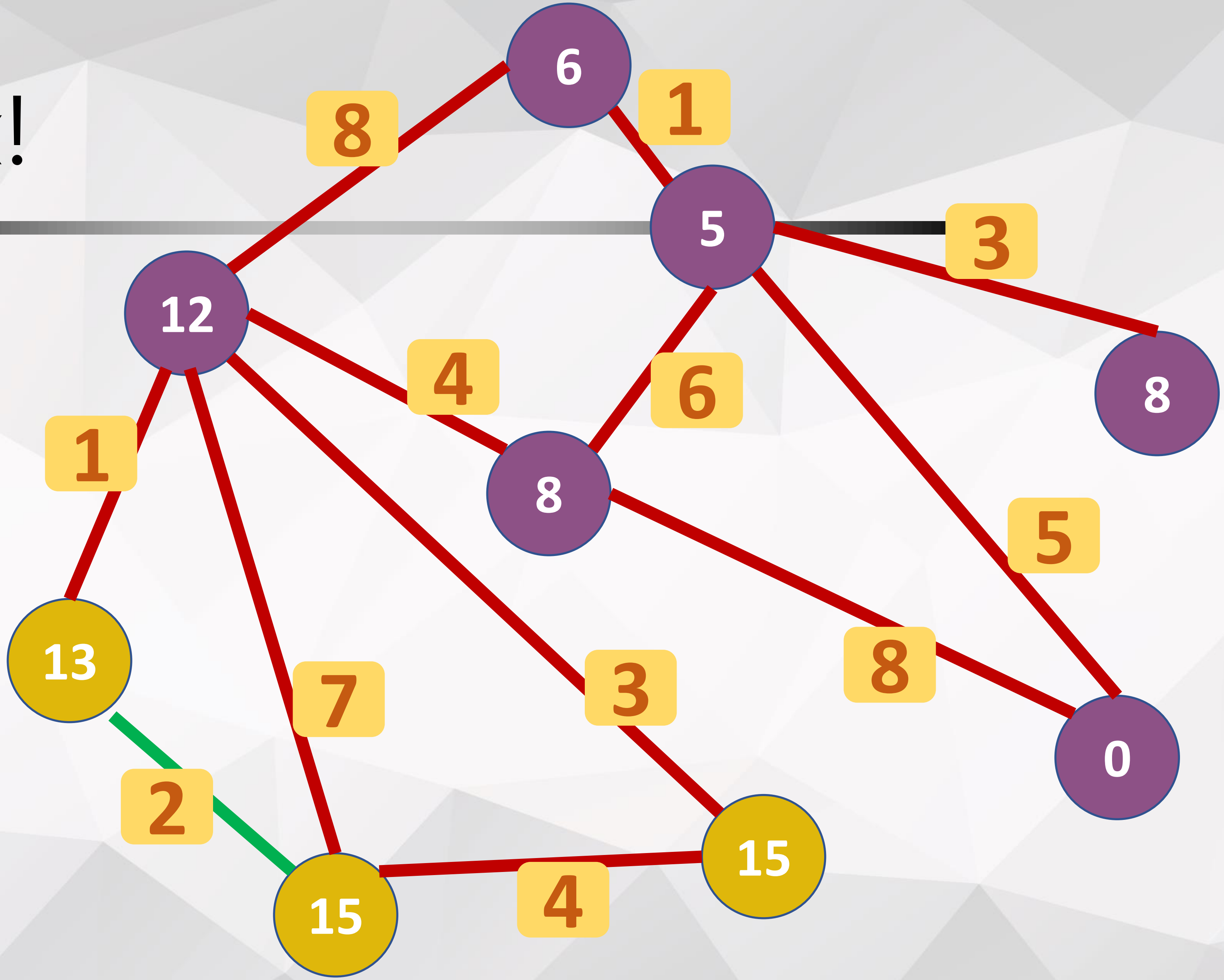
拜訪鄰點



Relax?



Relax!



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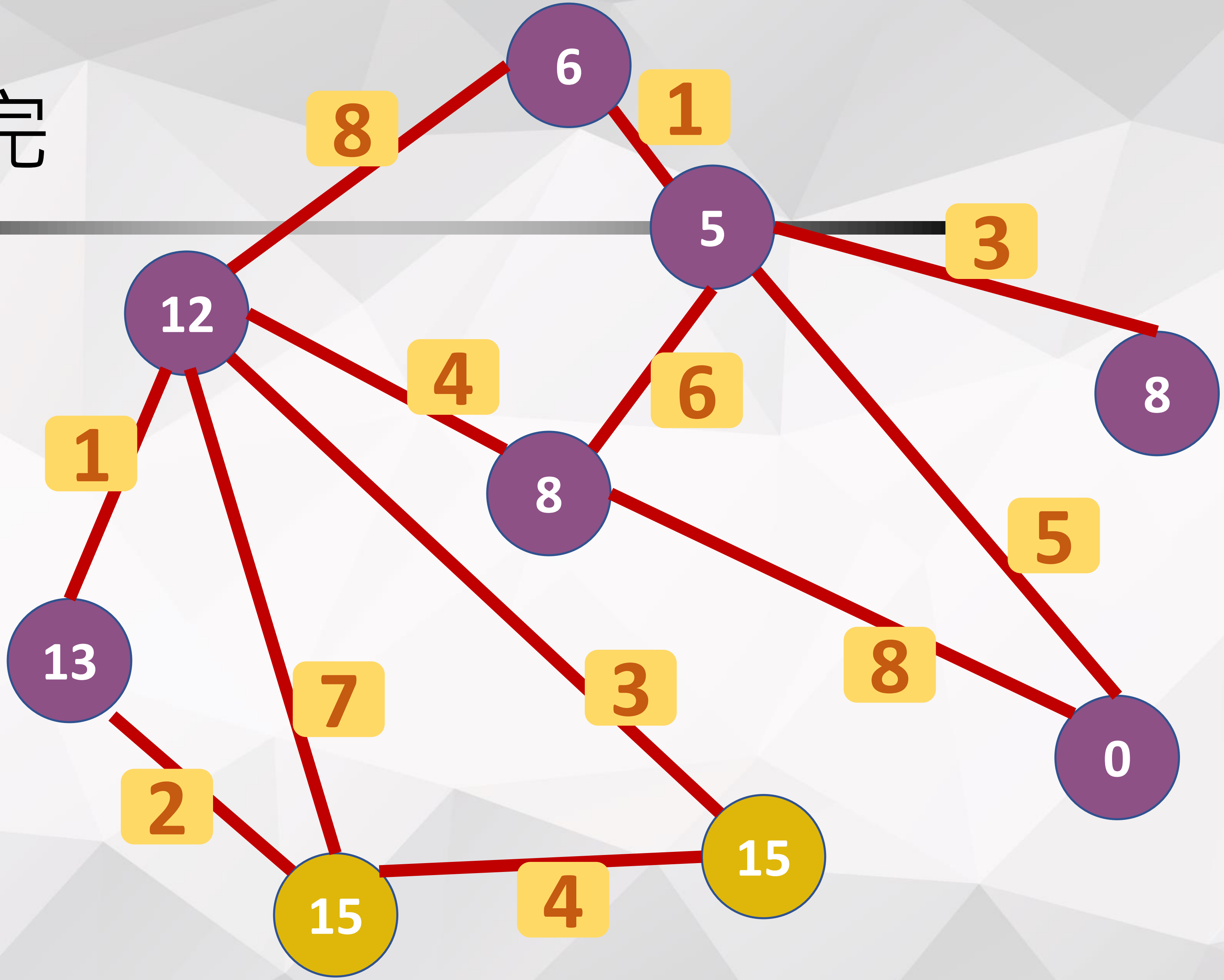
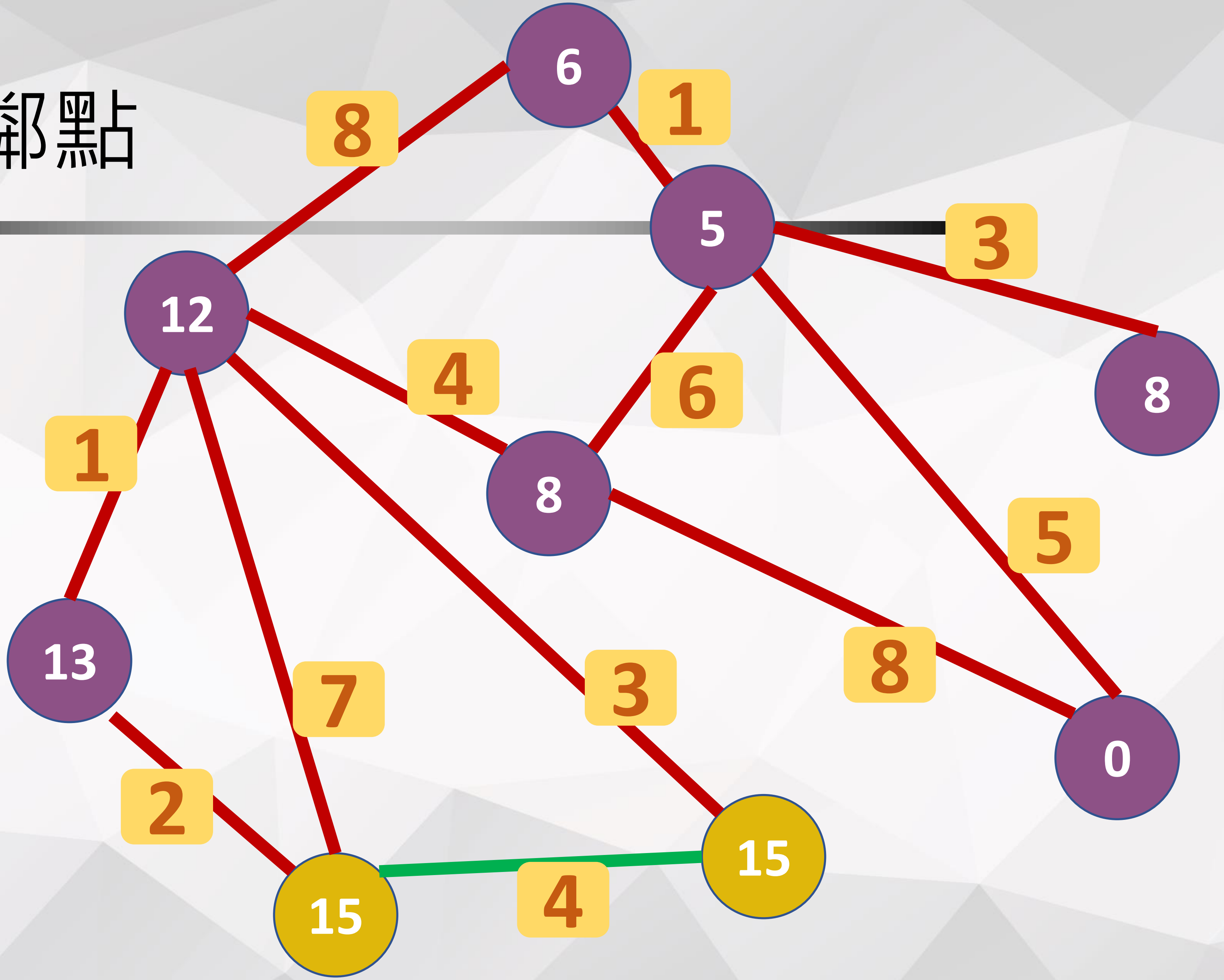
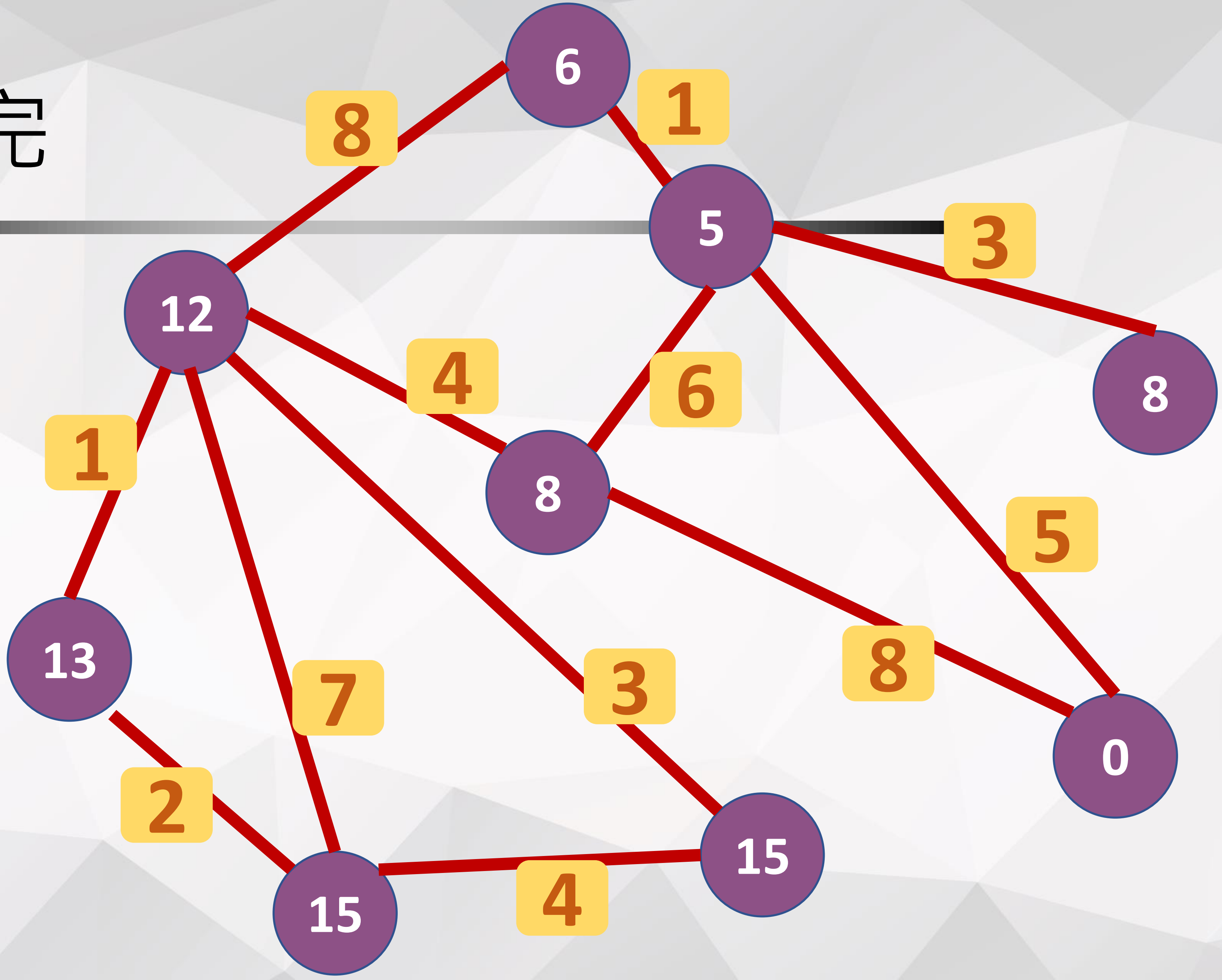


拜訪鄰點



拜訪完



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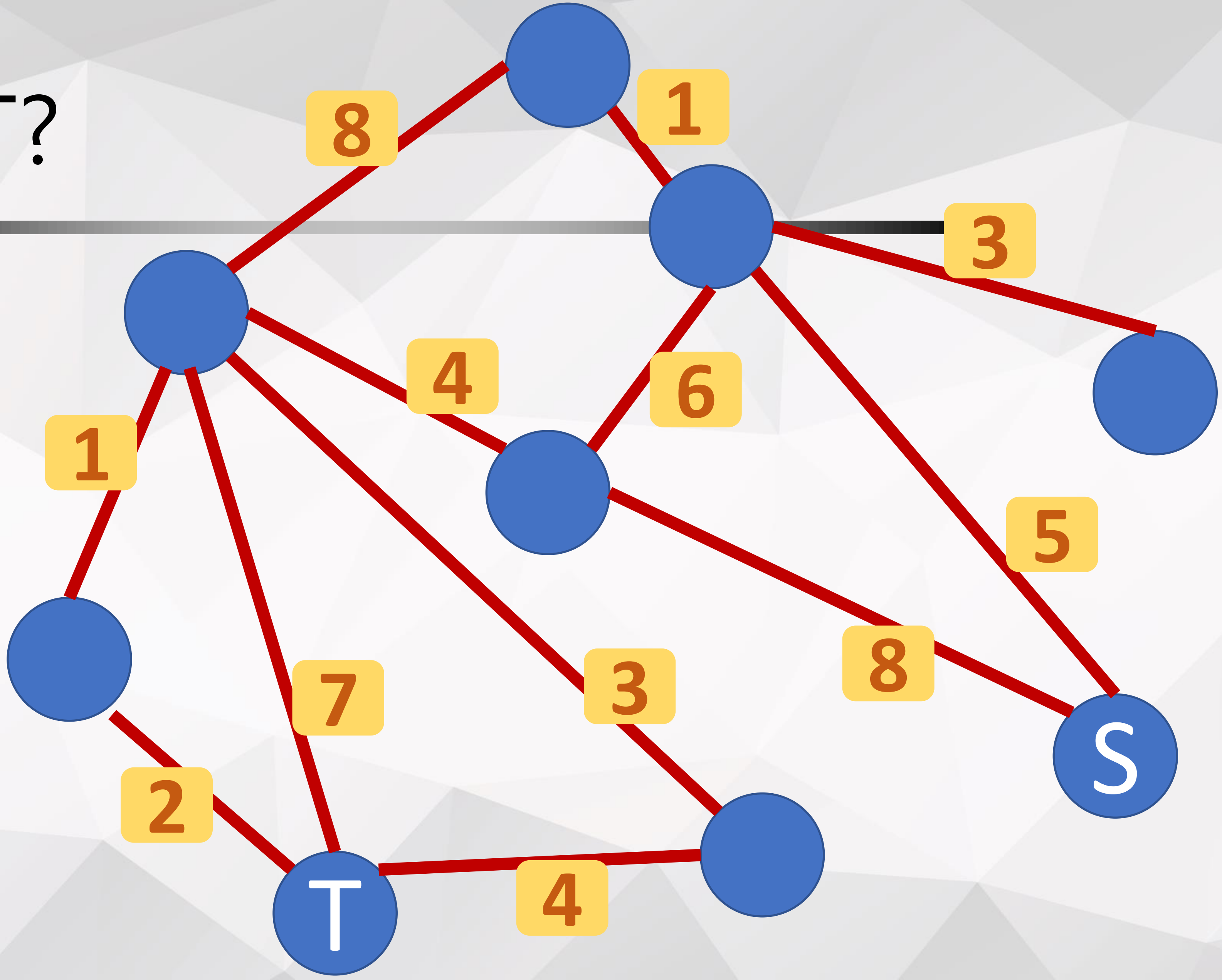

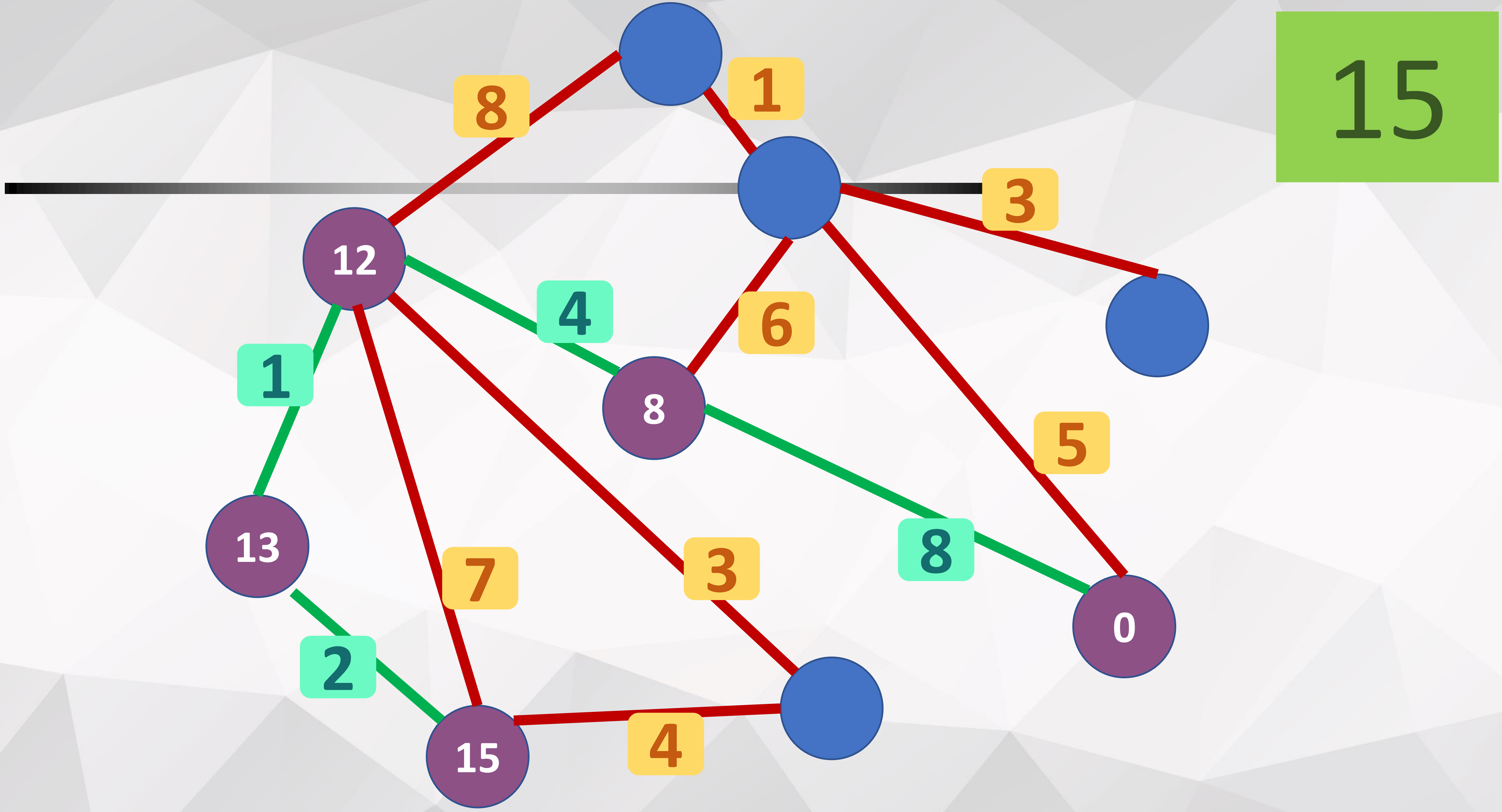
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Questions?

練習

- POJ 3255 Roadblocks

單源最短路徑

- Relaxation
- Dijkstra's algorithm
- Bellman-Ford's algorithm

Bellman-Ford's algorithm

Bellman-Ford 實作

```
vector<edge> E;
    :
    .

/* 假設輸入完邊的資訊了 */
```

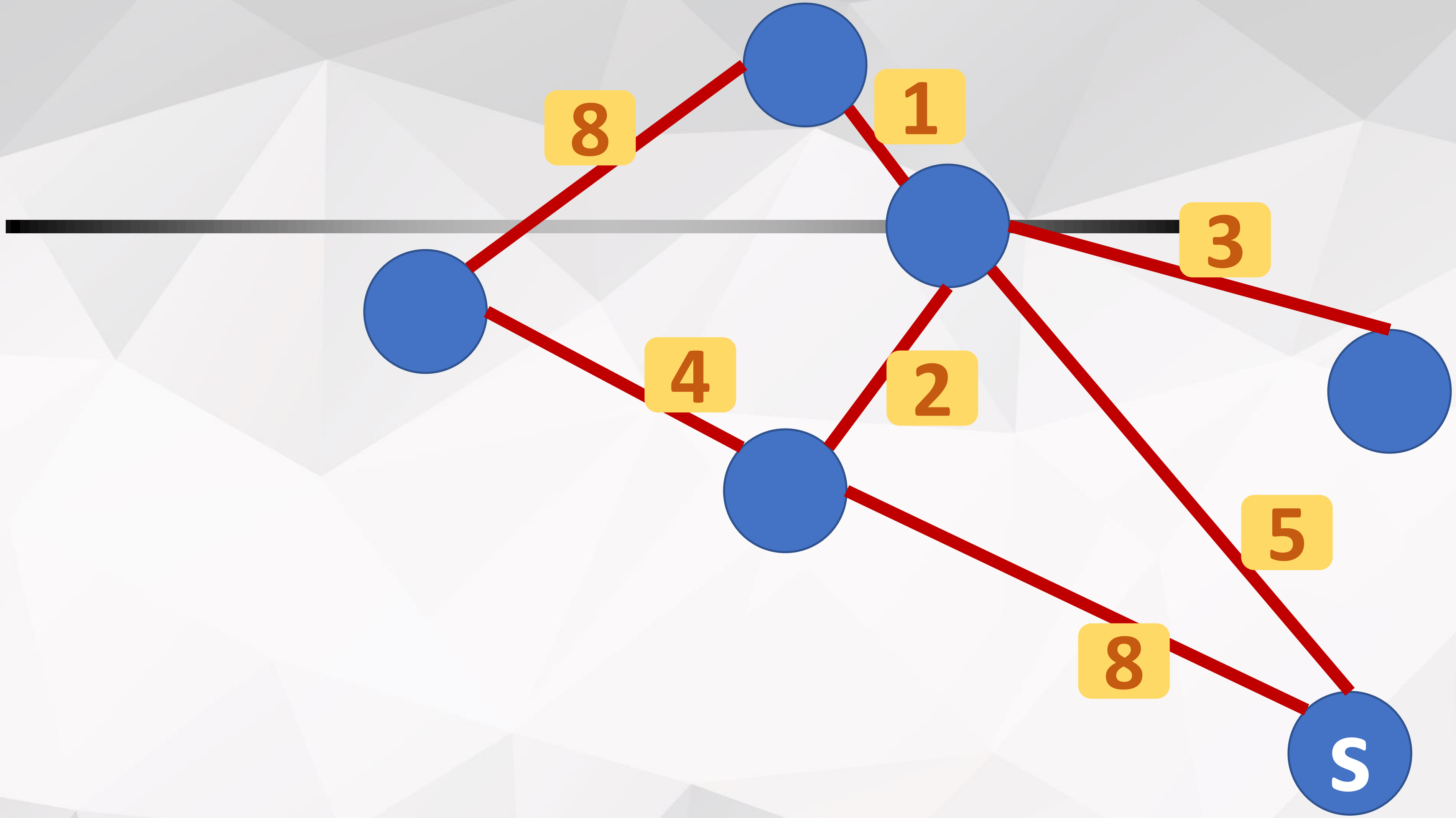
Bellman-Ford 實作

```
memset(s, 0x3f, sizeof(s)); // 初始無限大

s[source] = 0;
```

Bellman-Ford 實作

```
for (int i = 0; i < V.size()-1; i++)
   for (edge e: E)
      s[e.v] = min(s[e.v], s[e.u] + e.w);
```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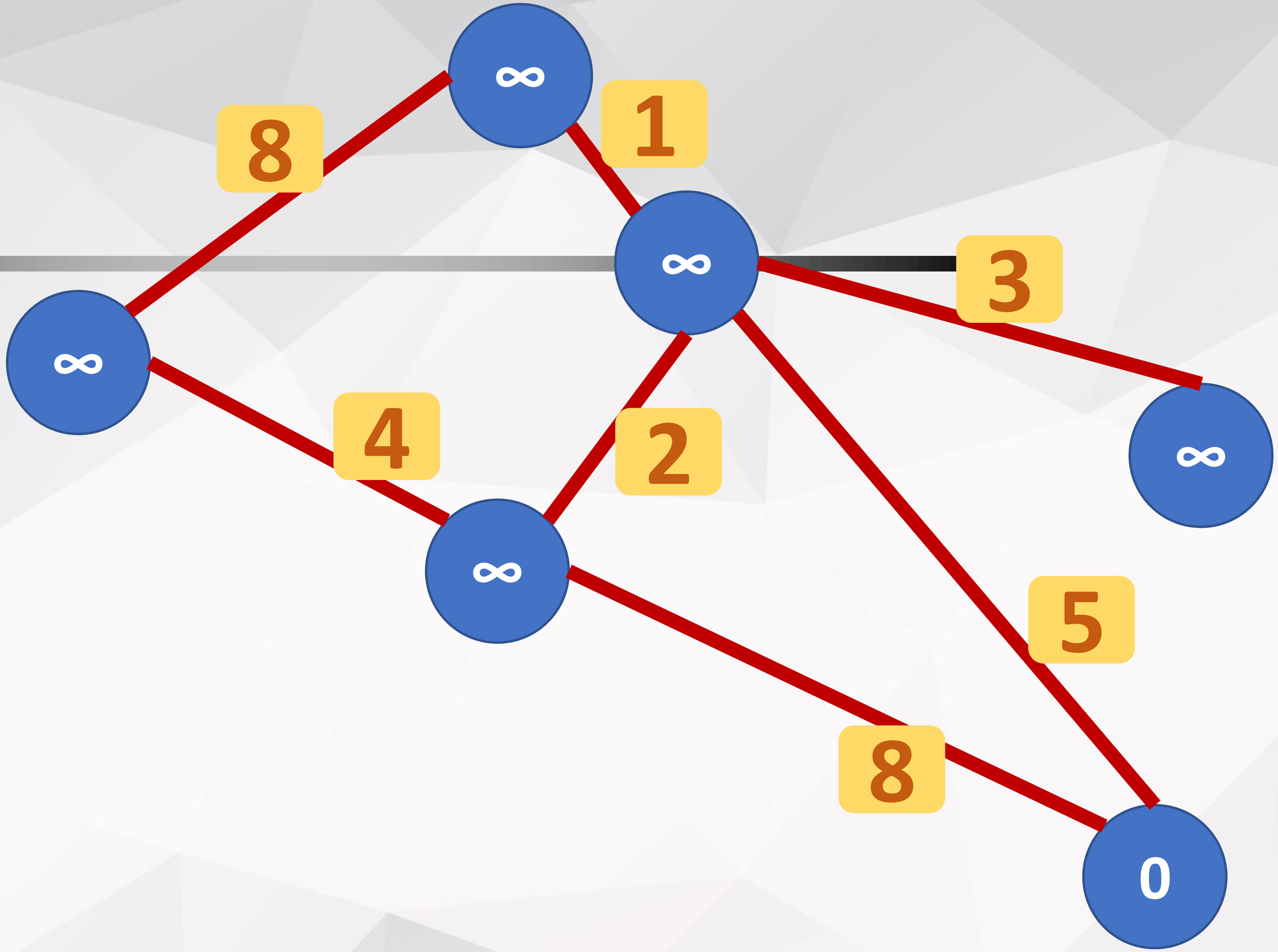

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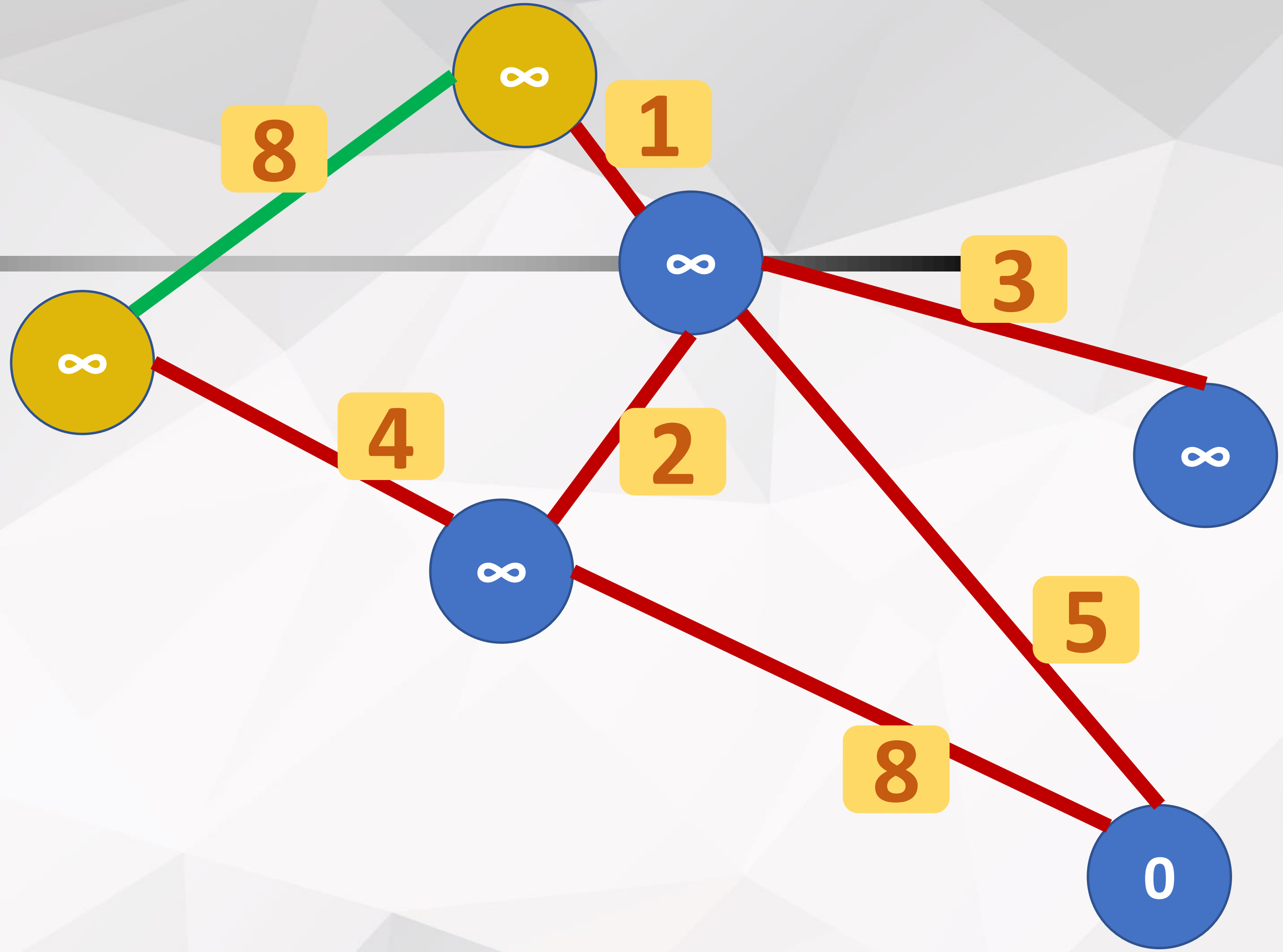

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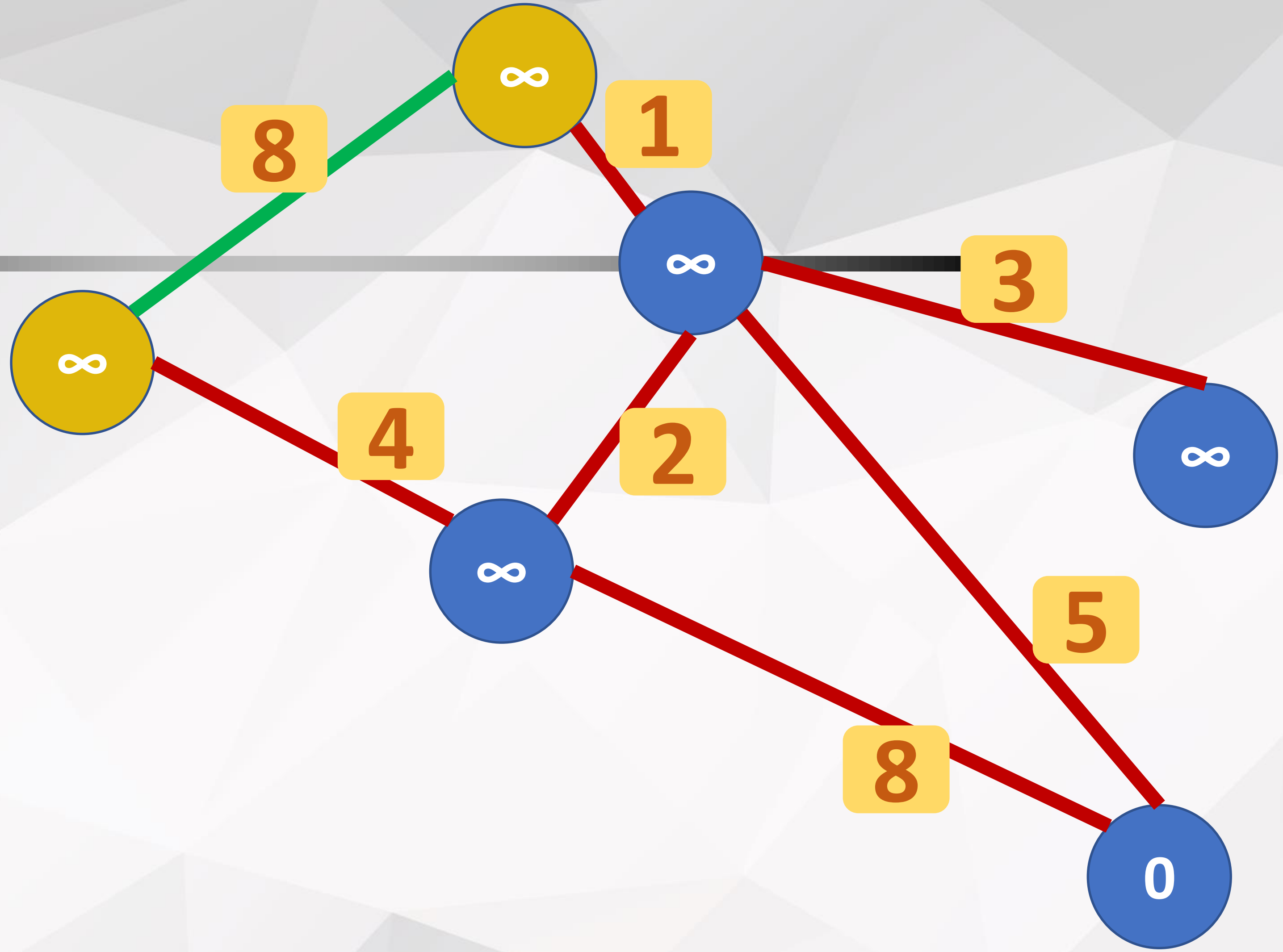

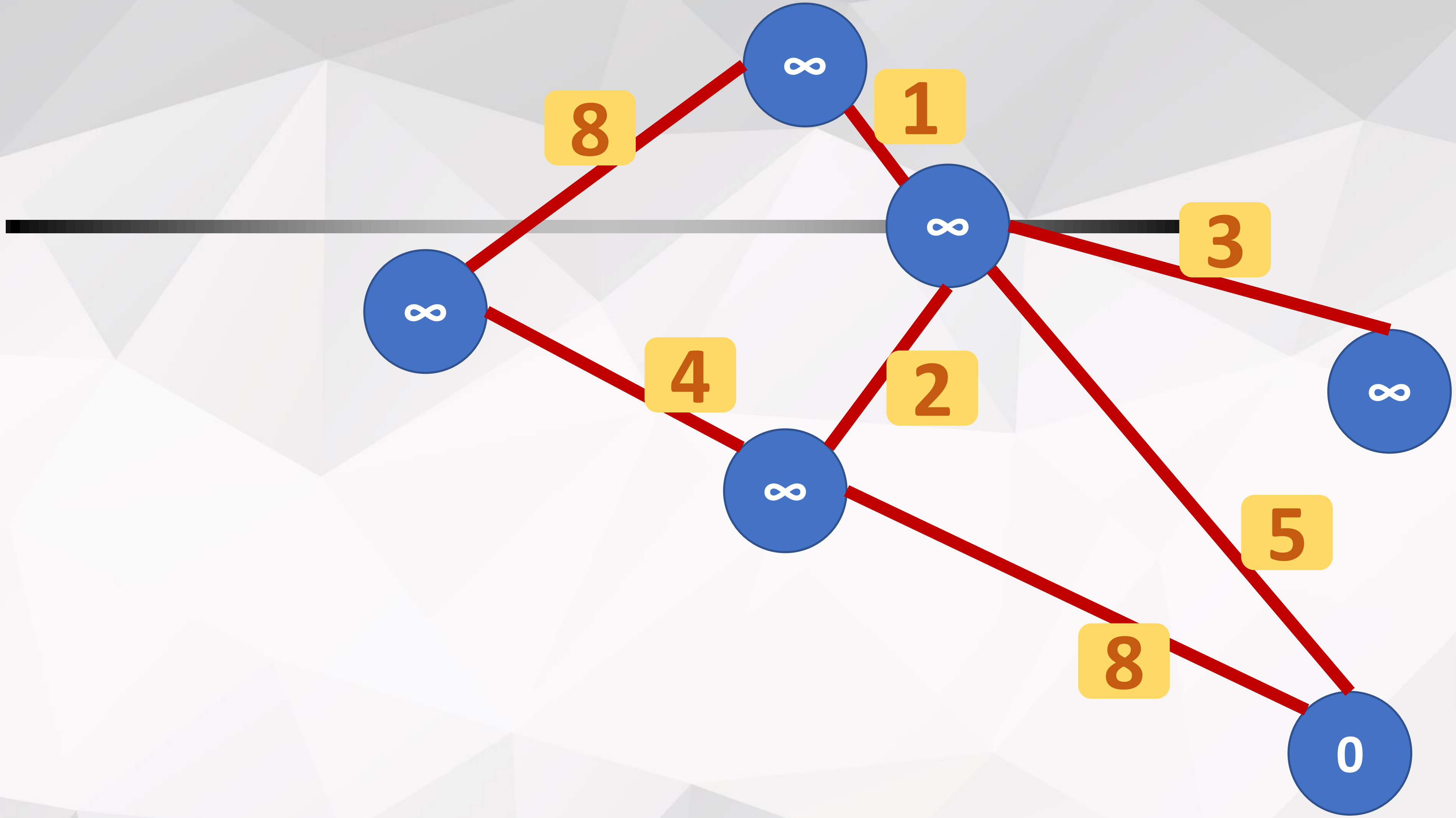
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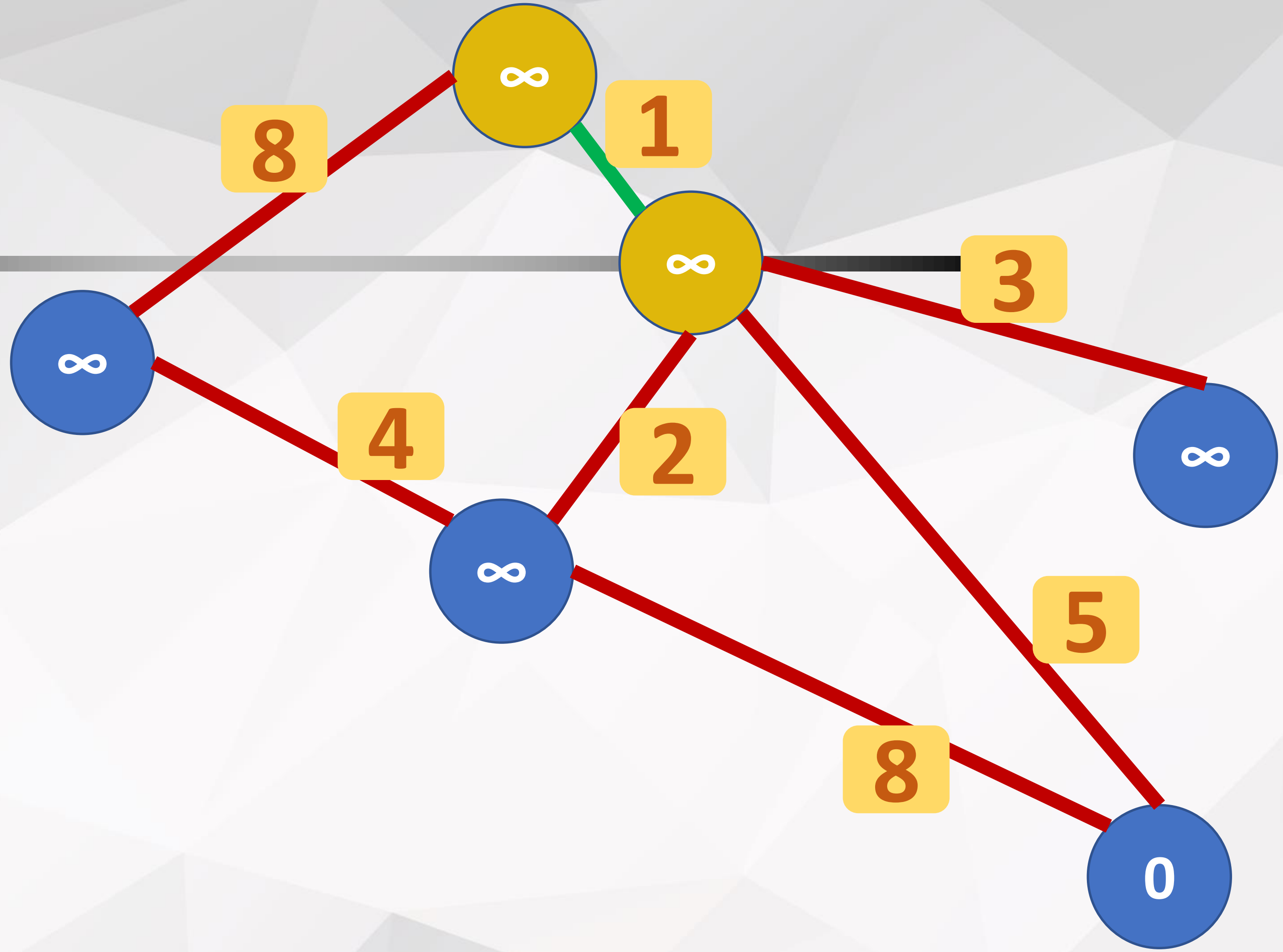

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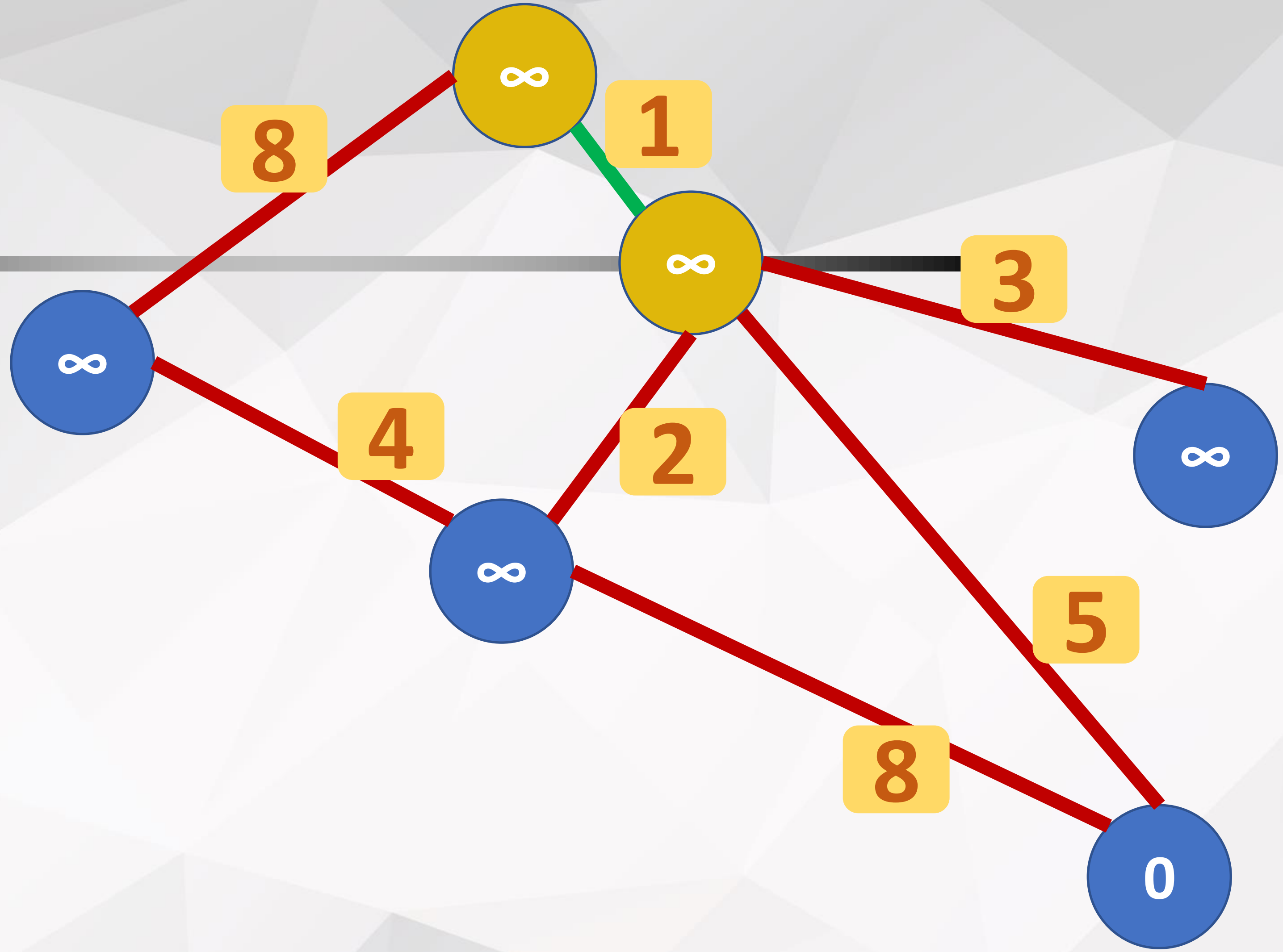

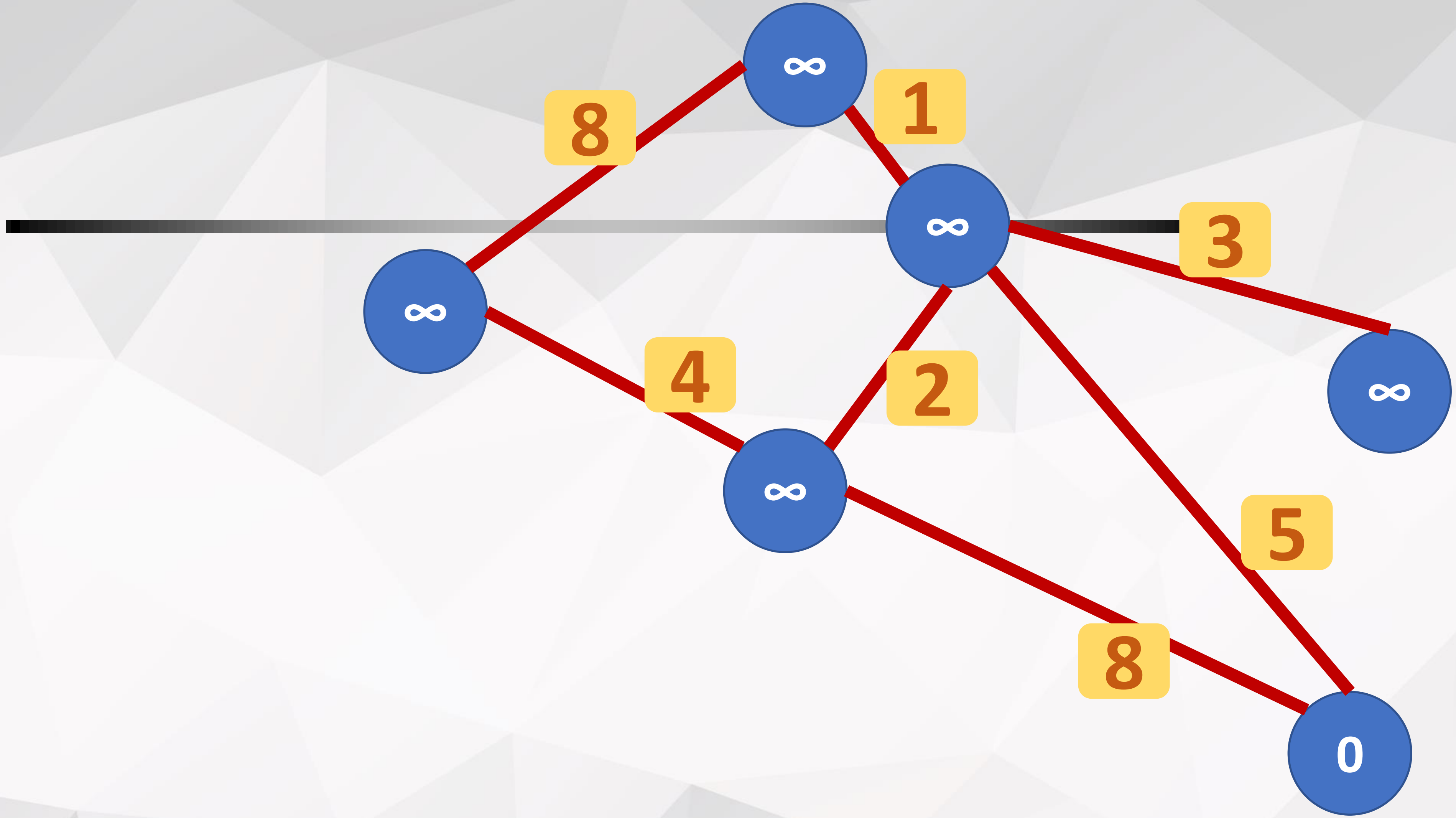
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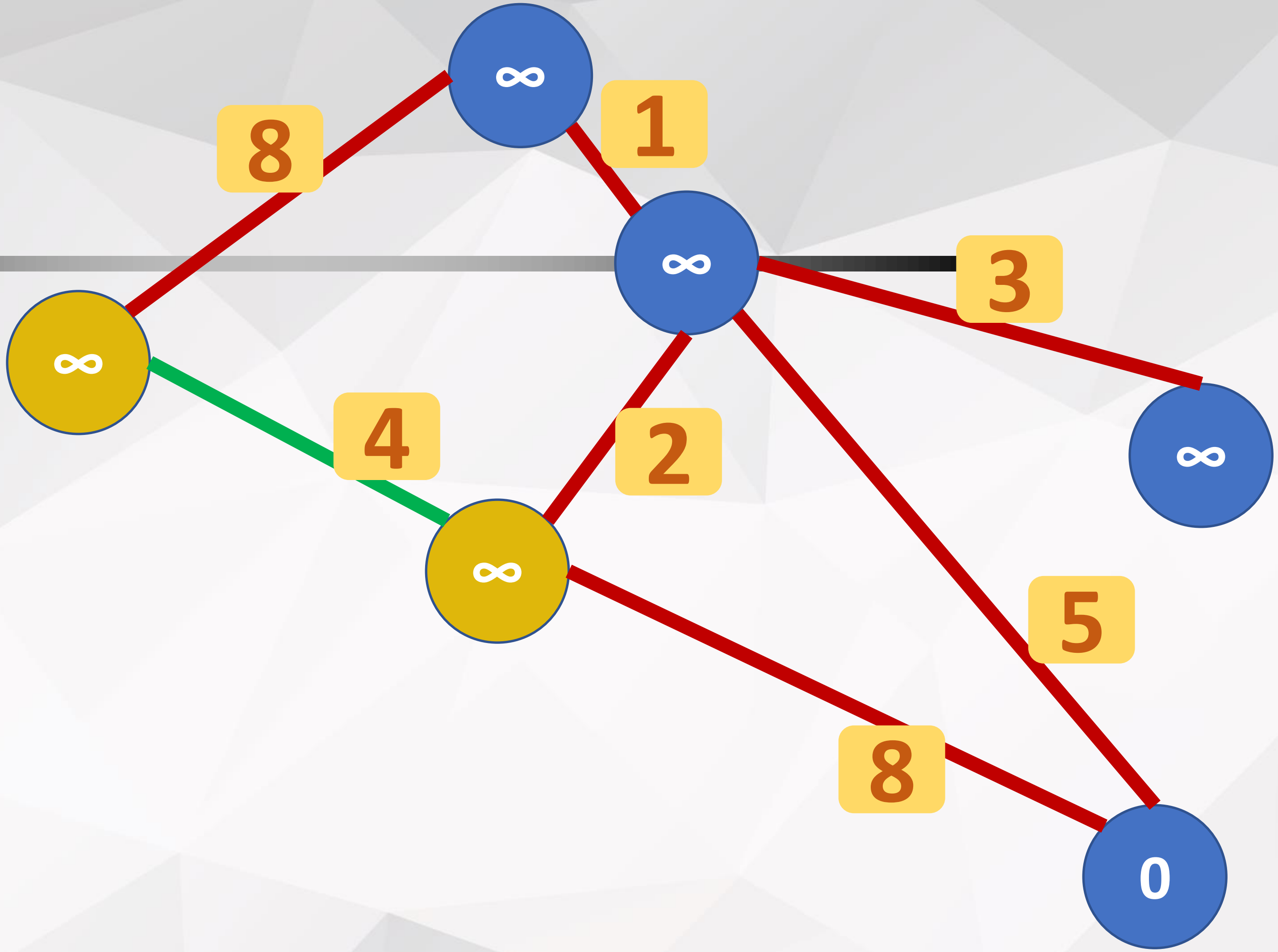

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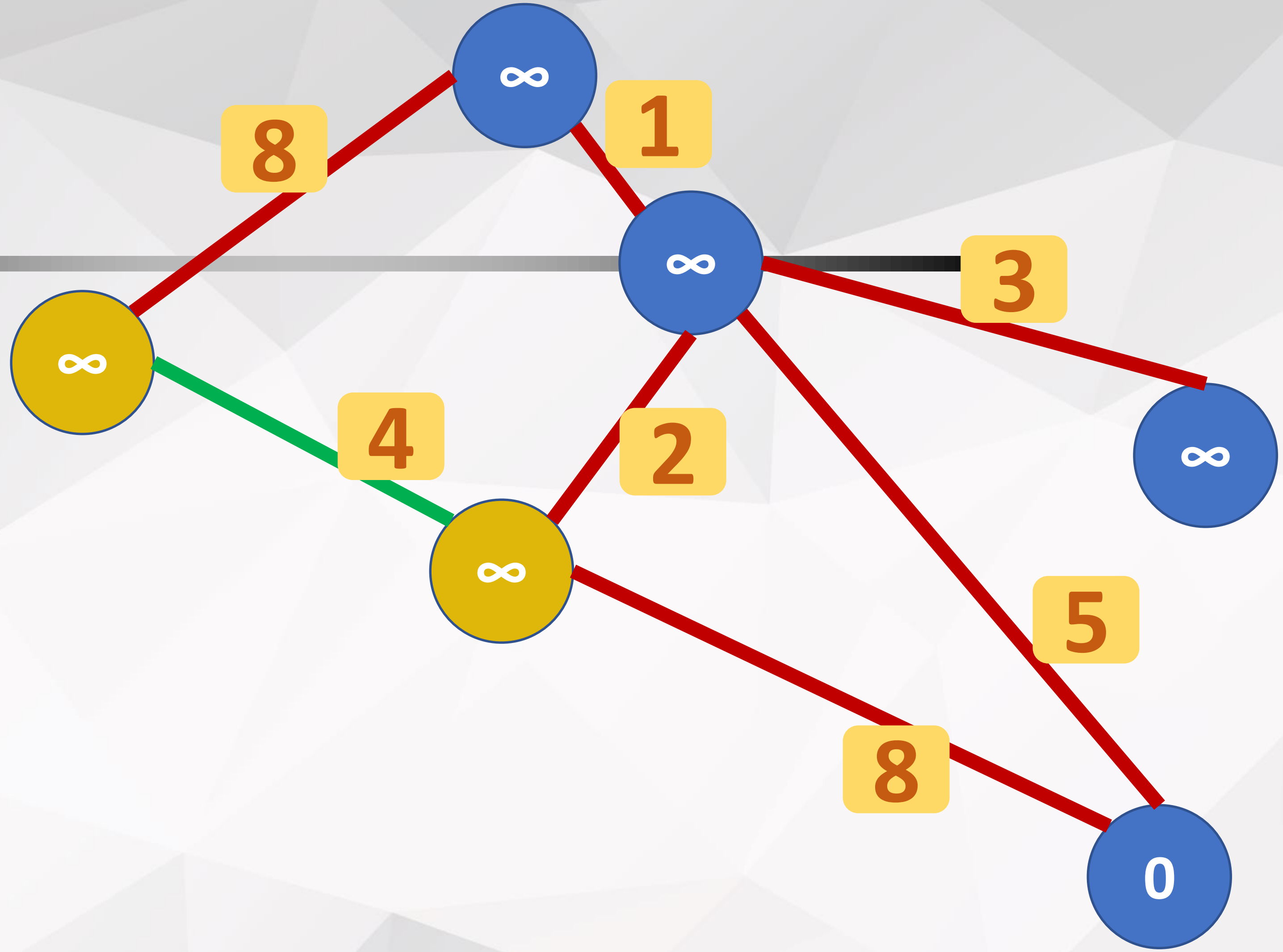

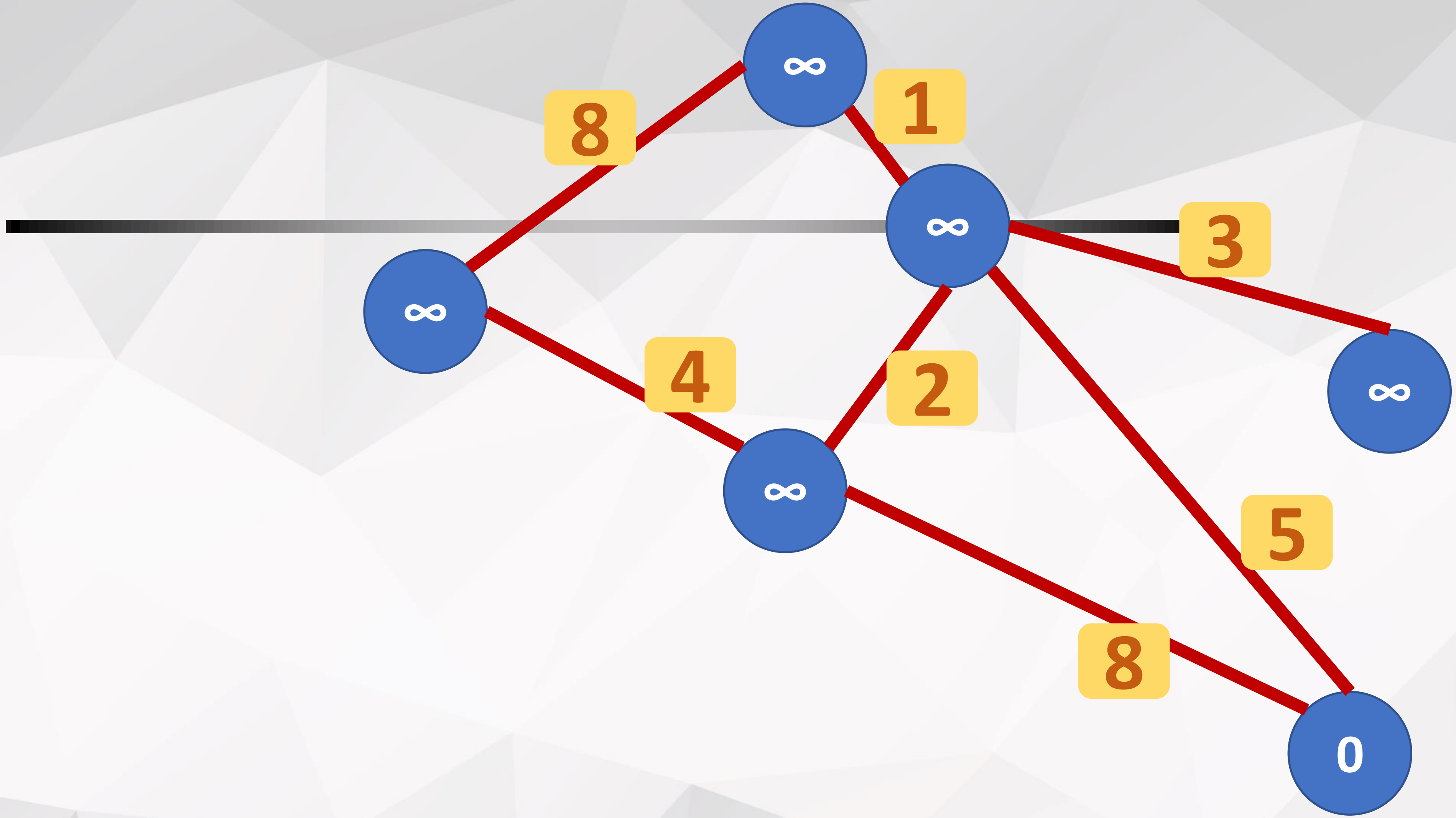
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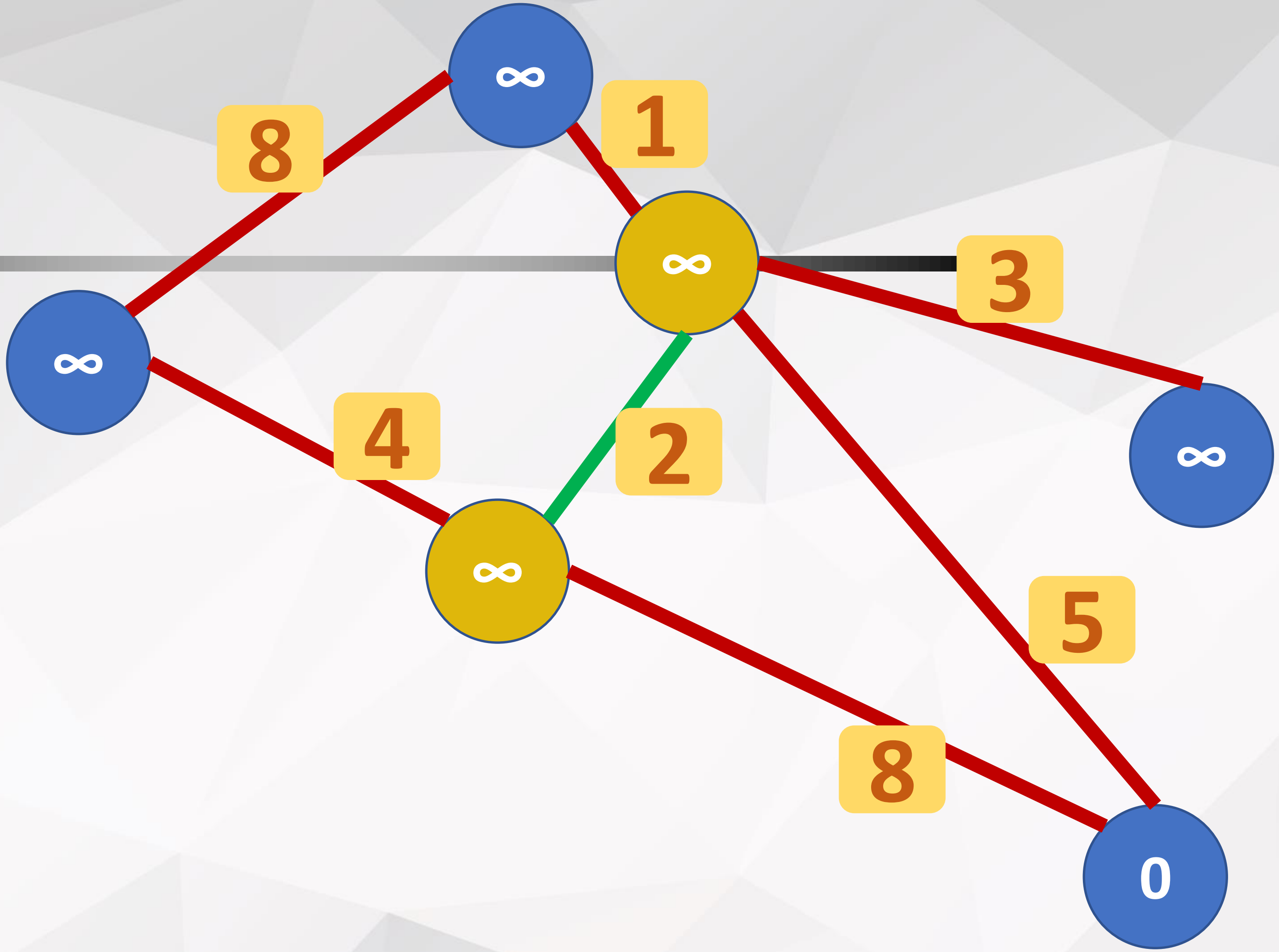

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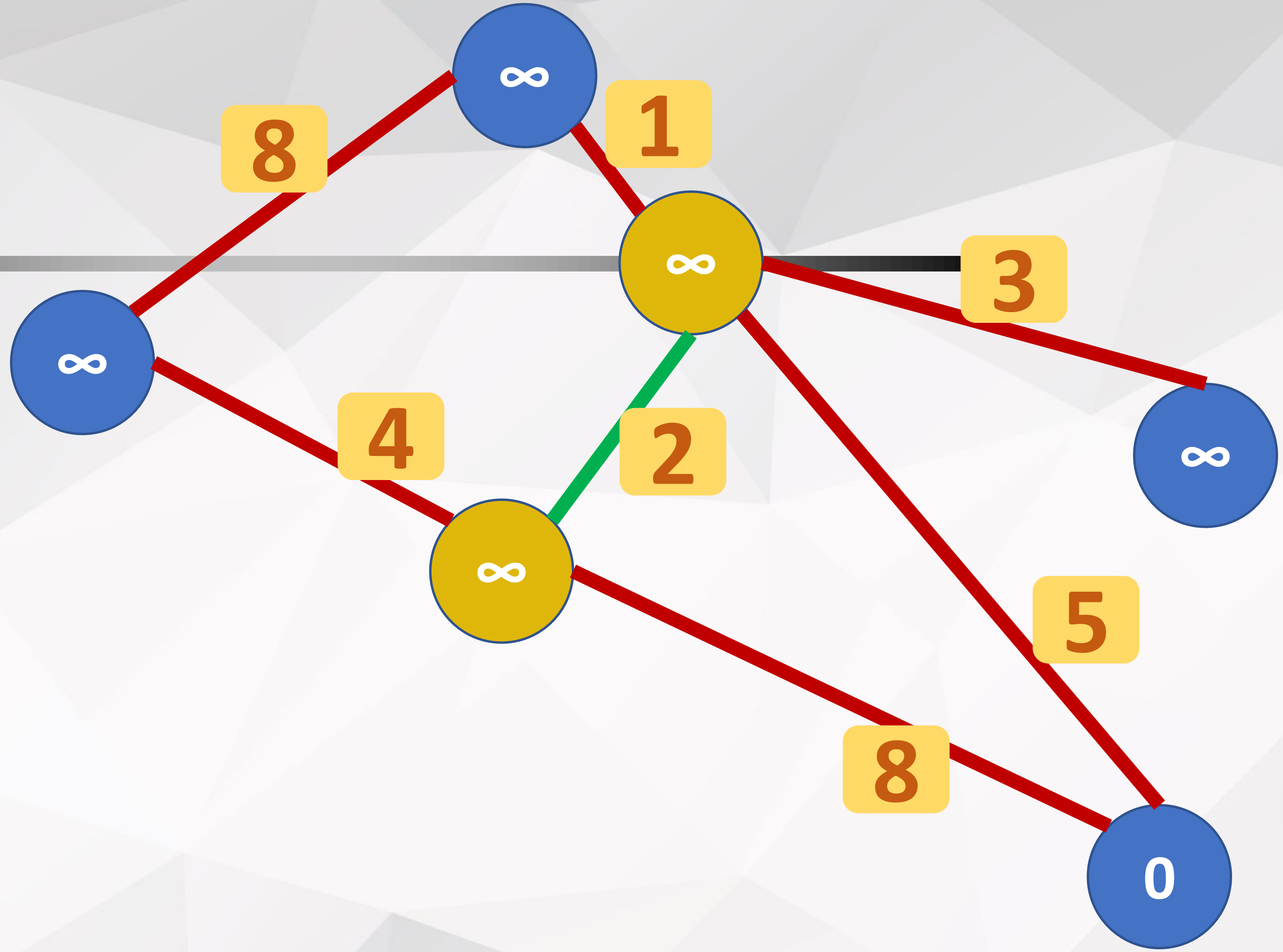

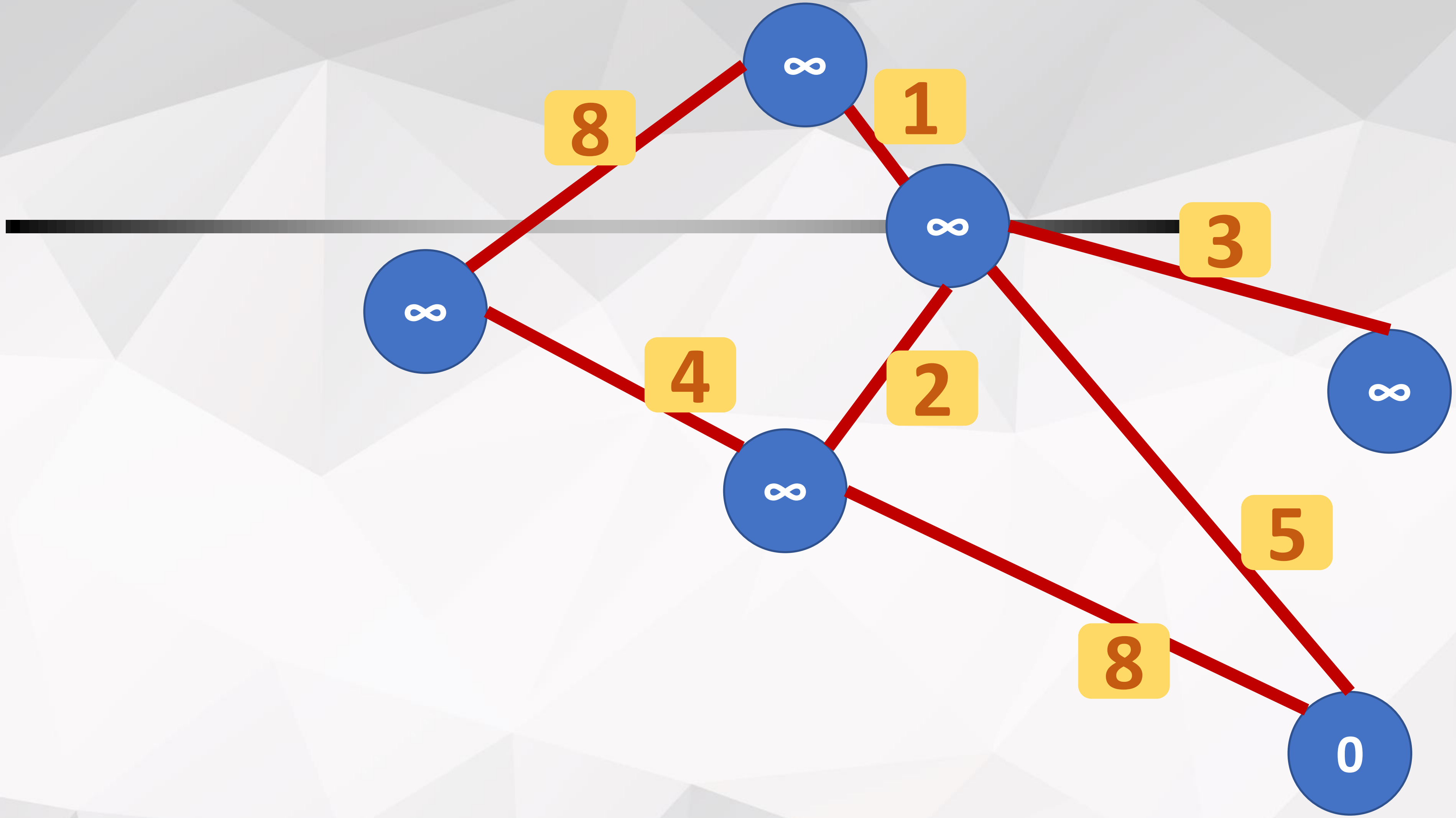
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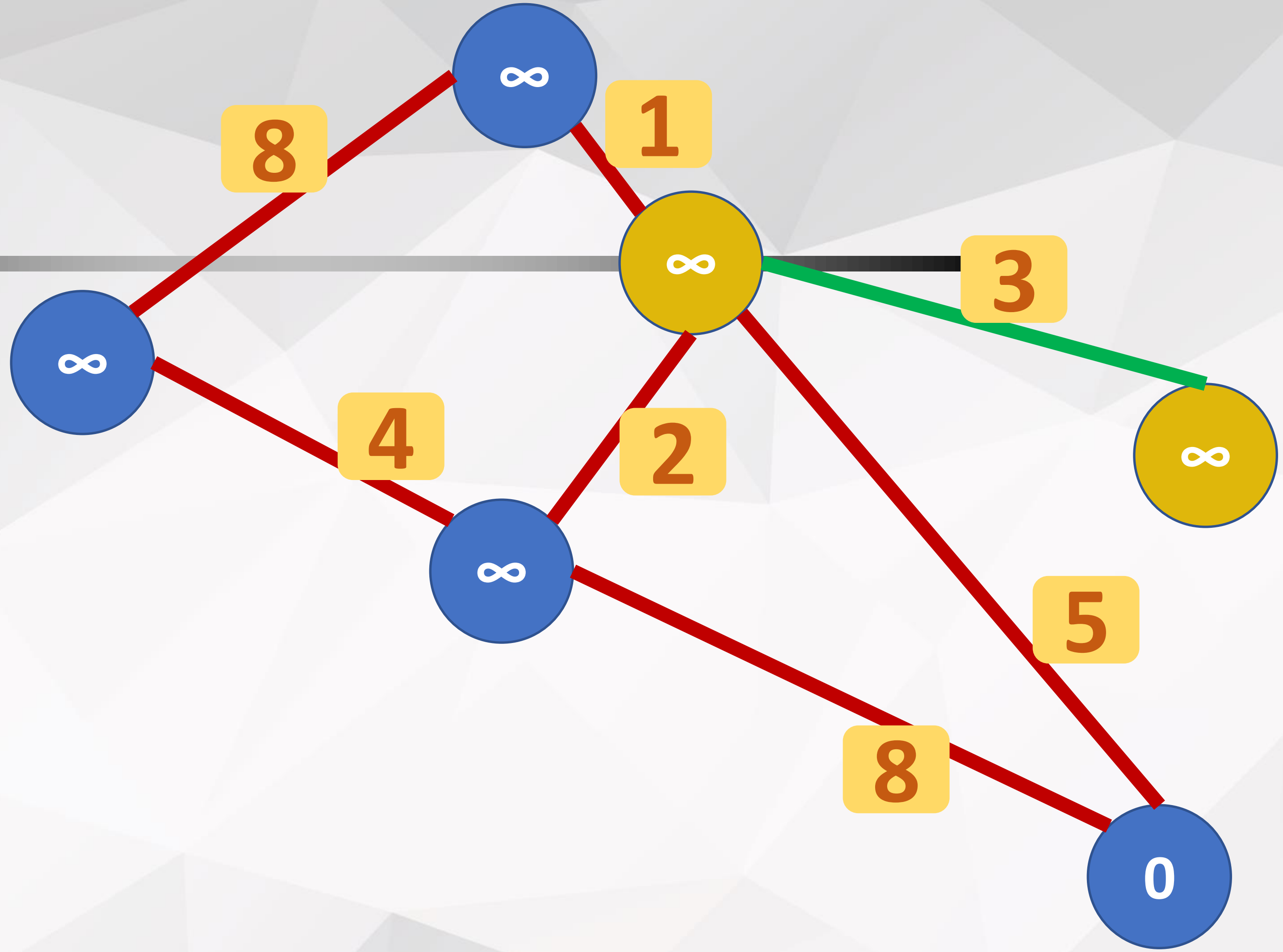

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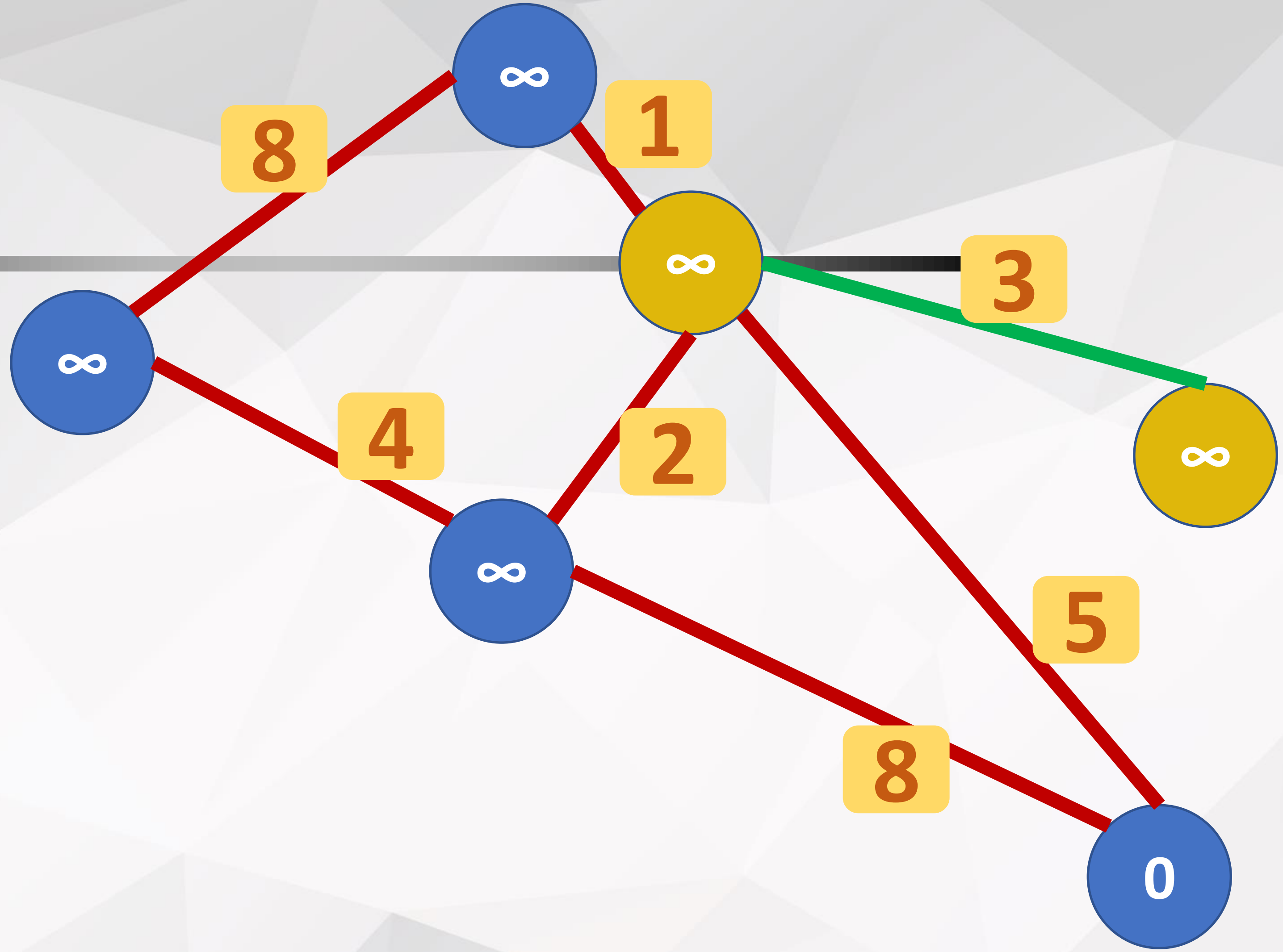

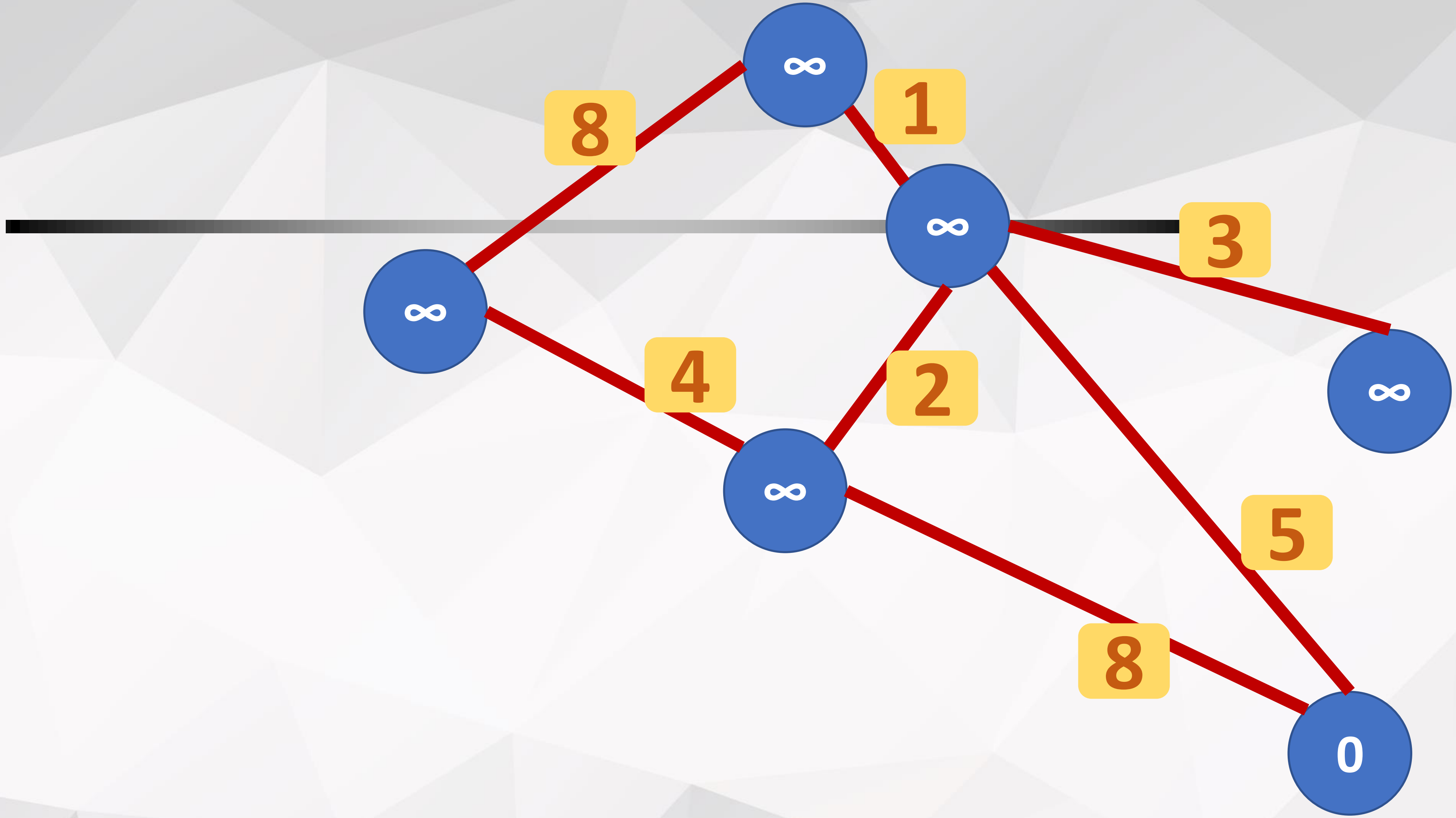
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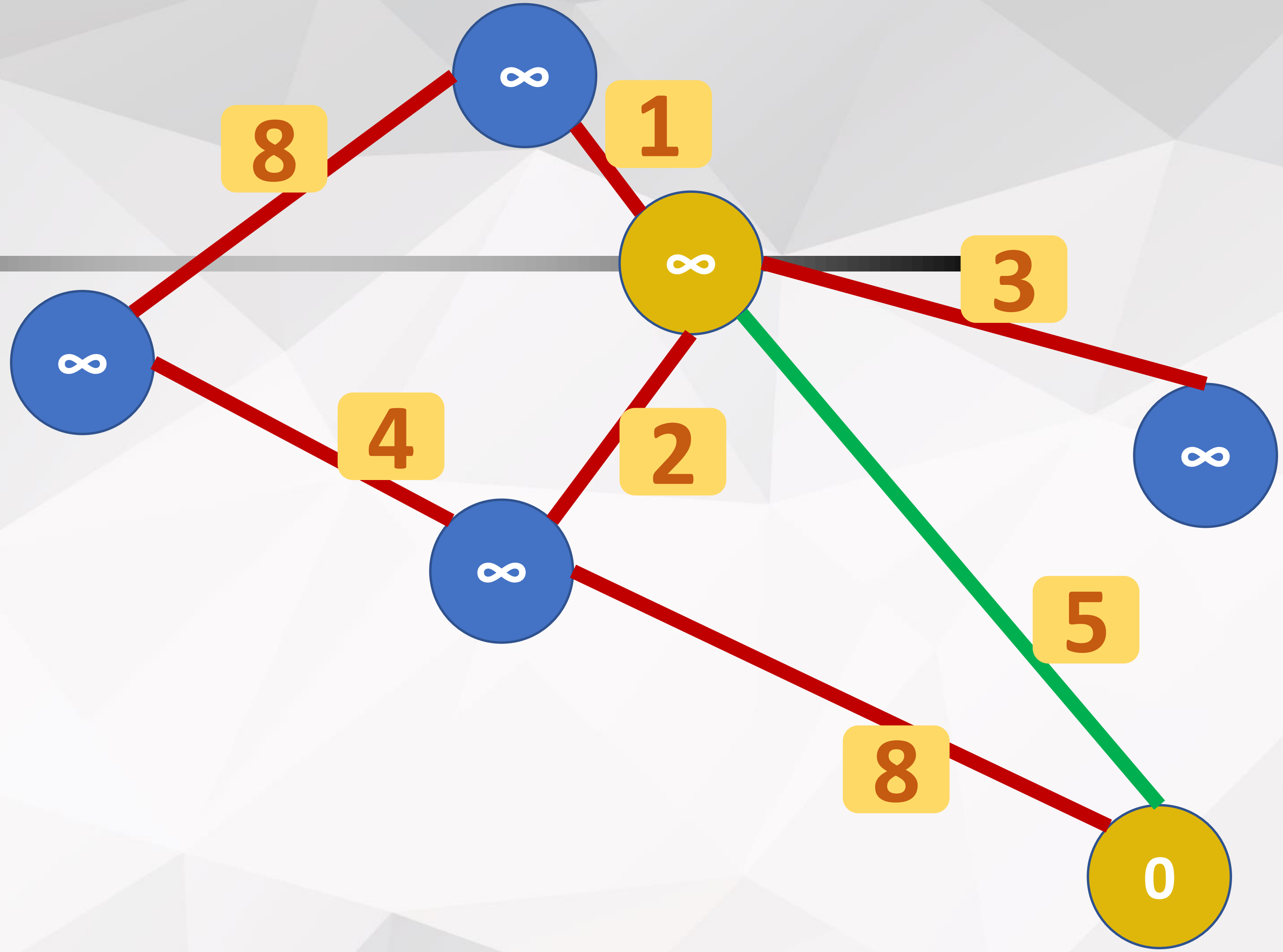

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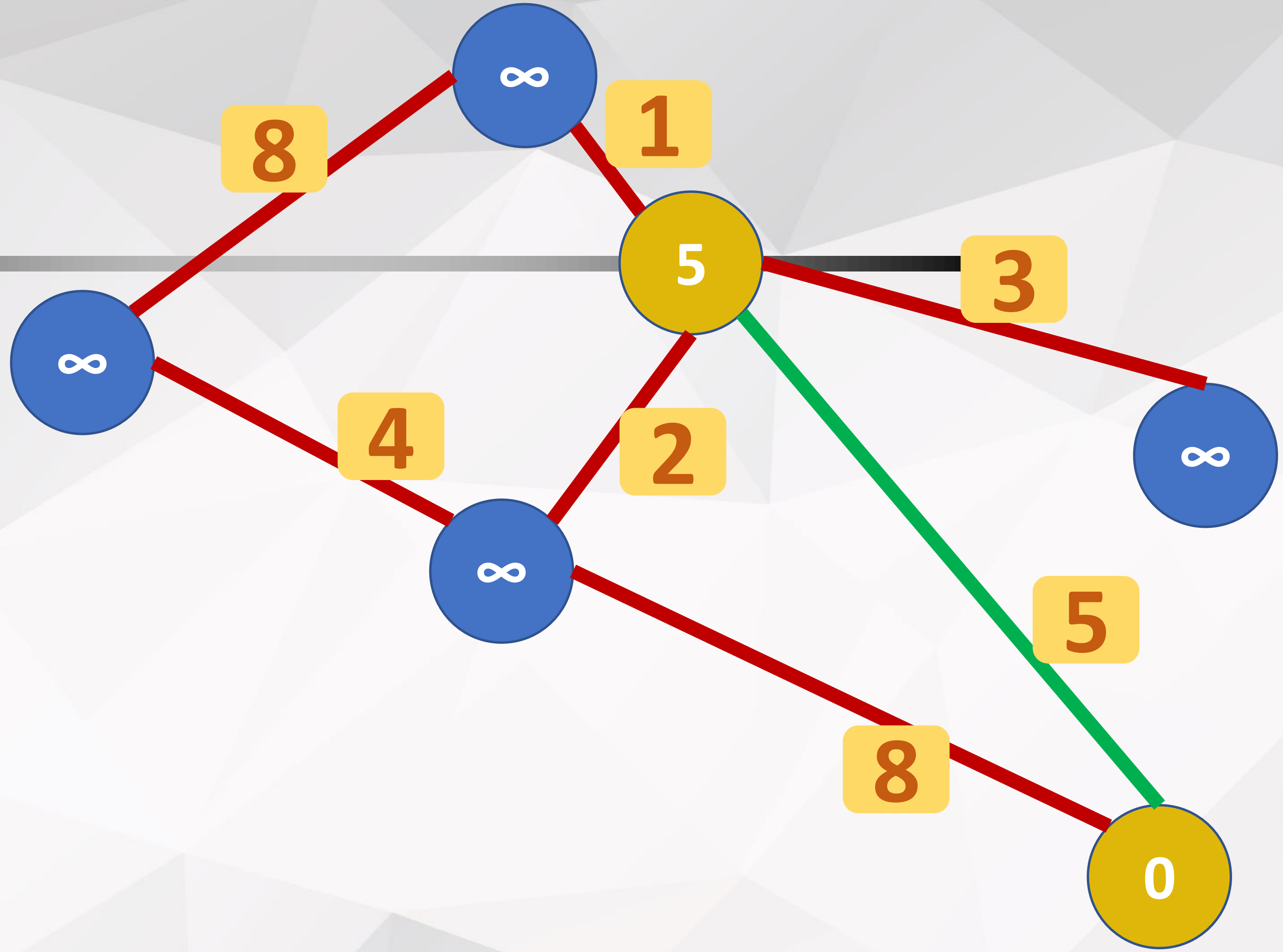

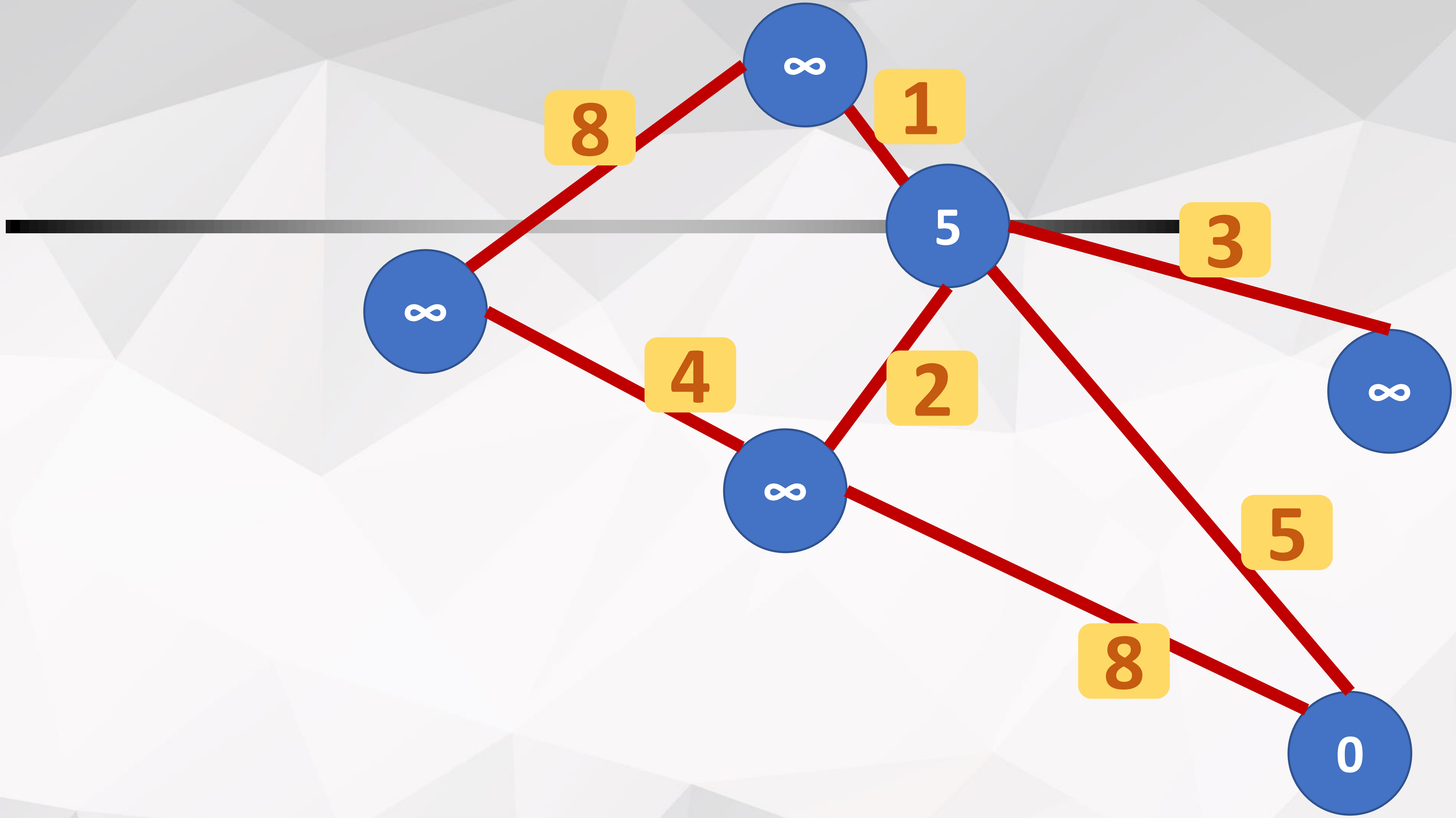
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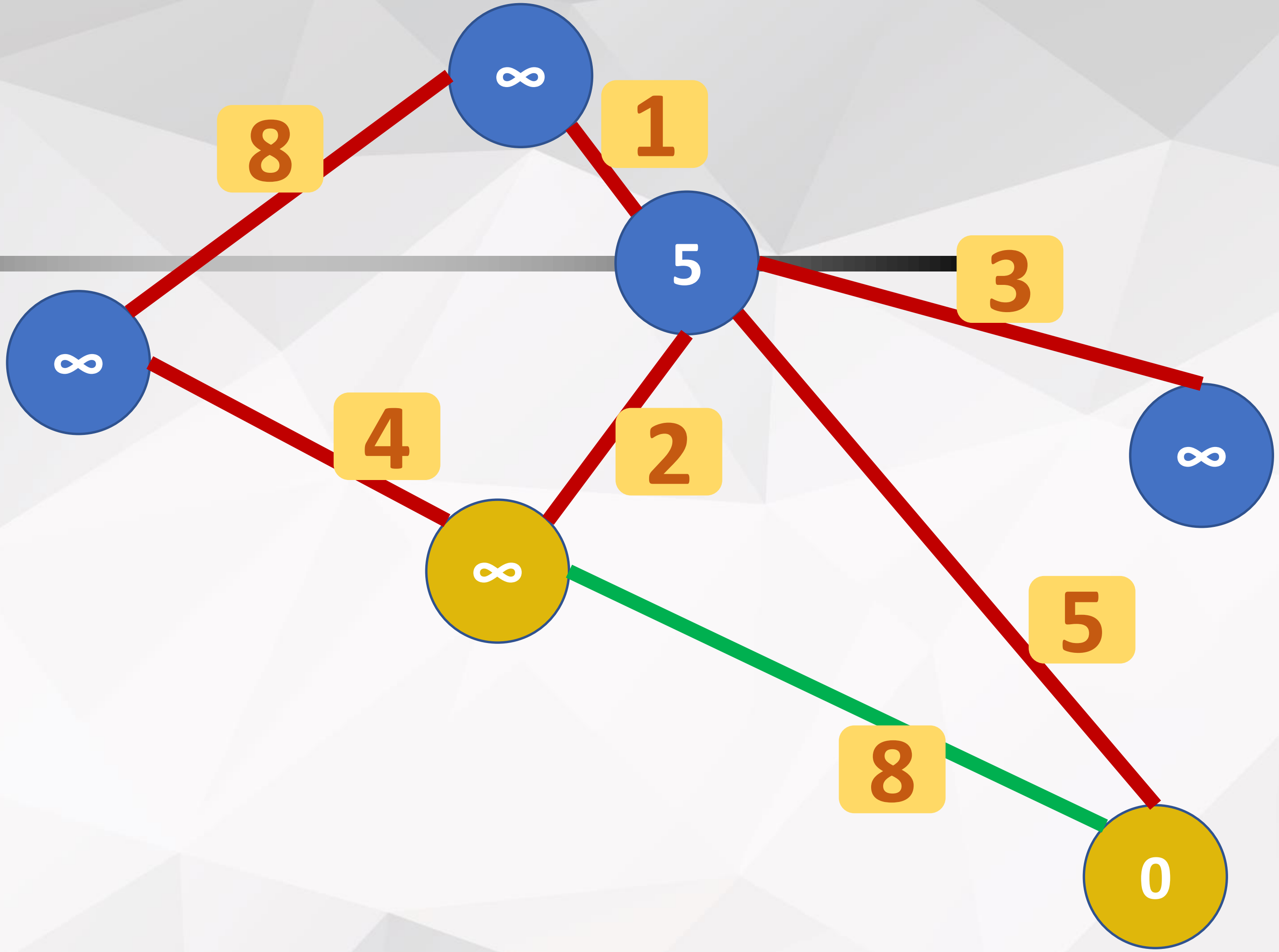

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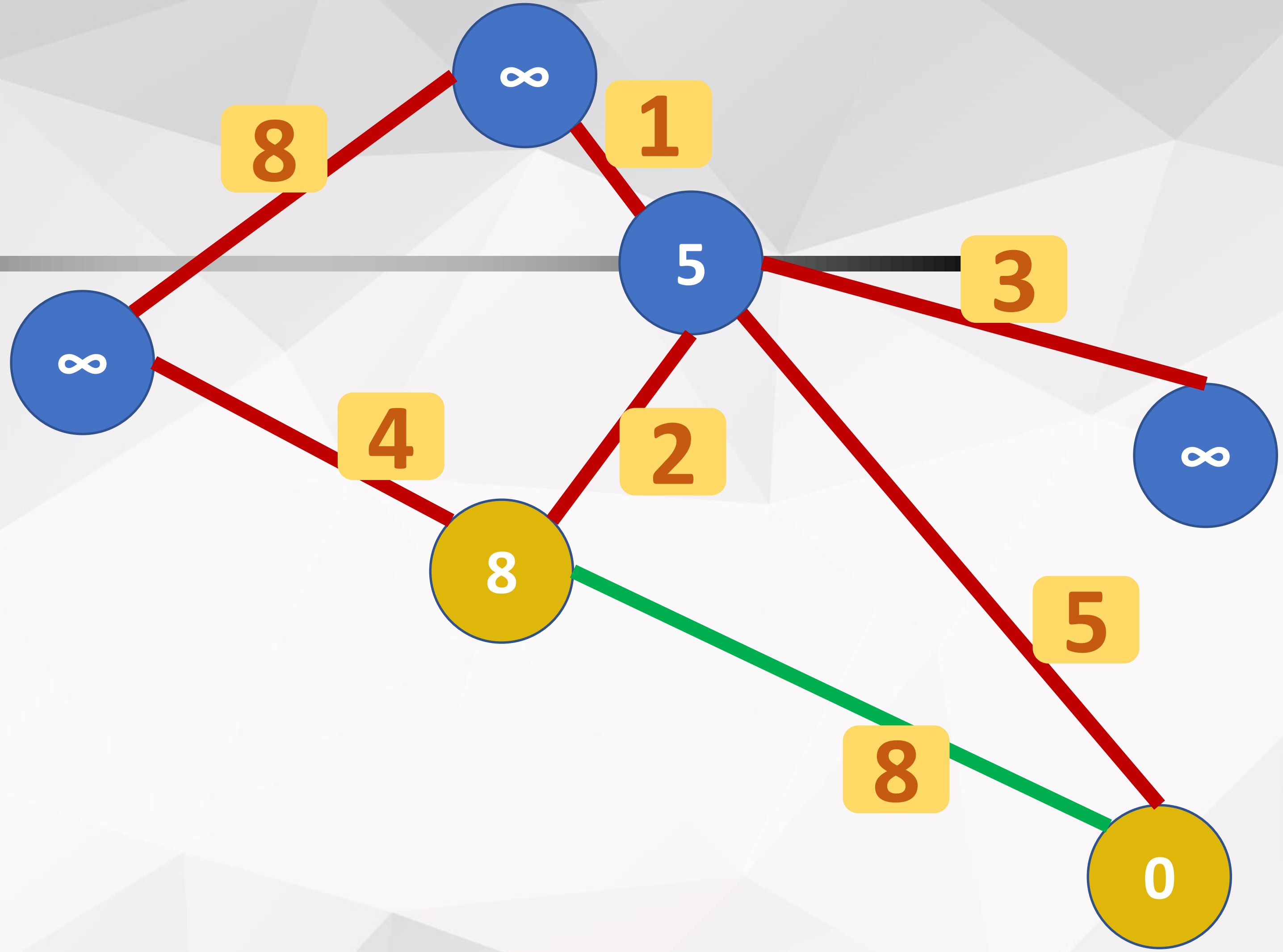

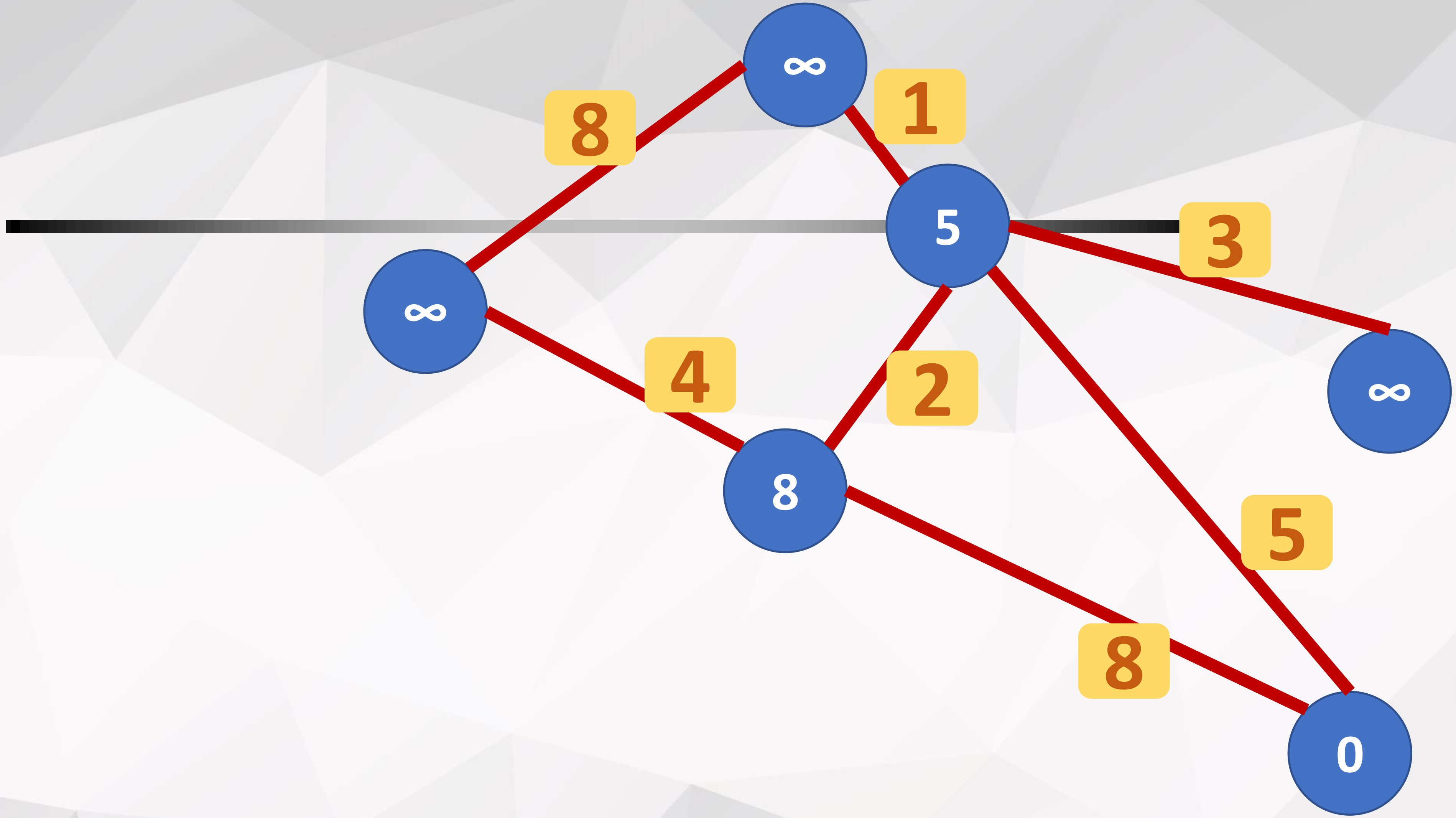
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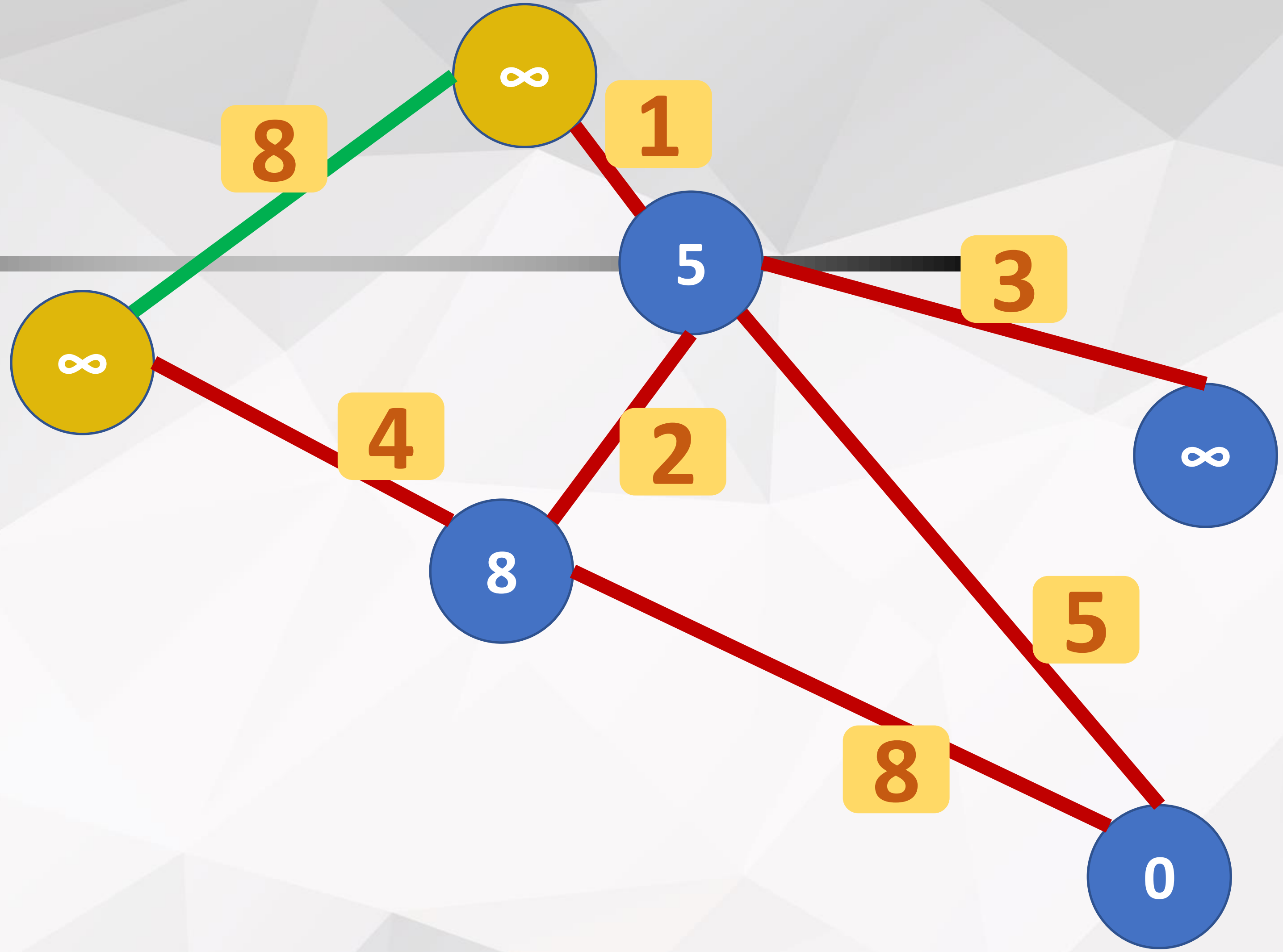

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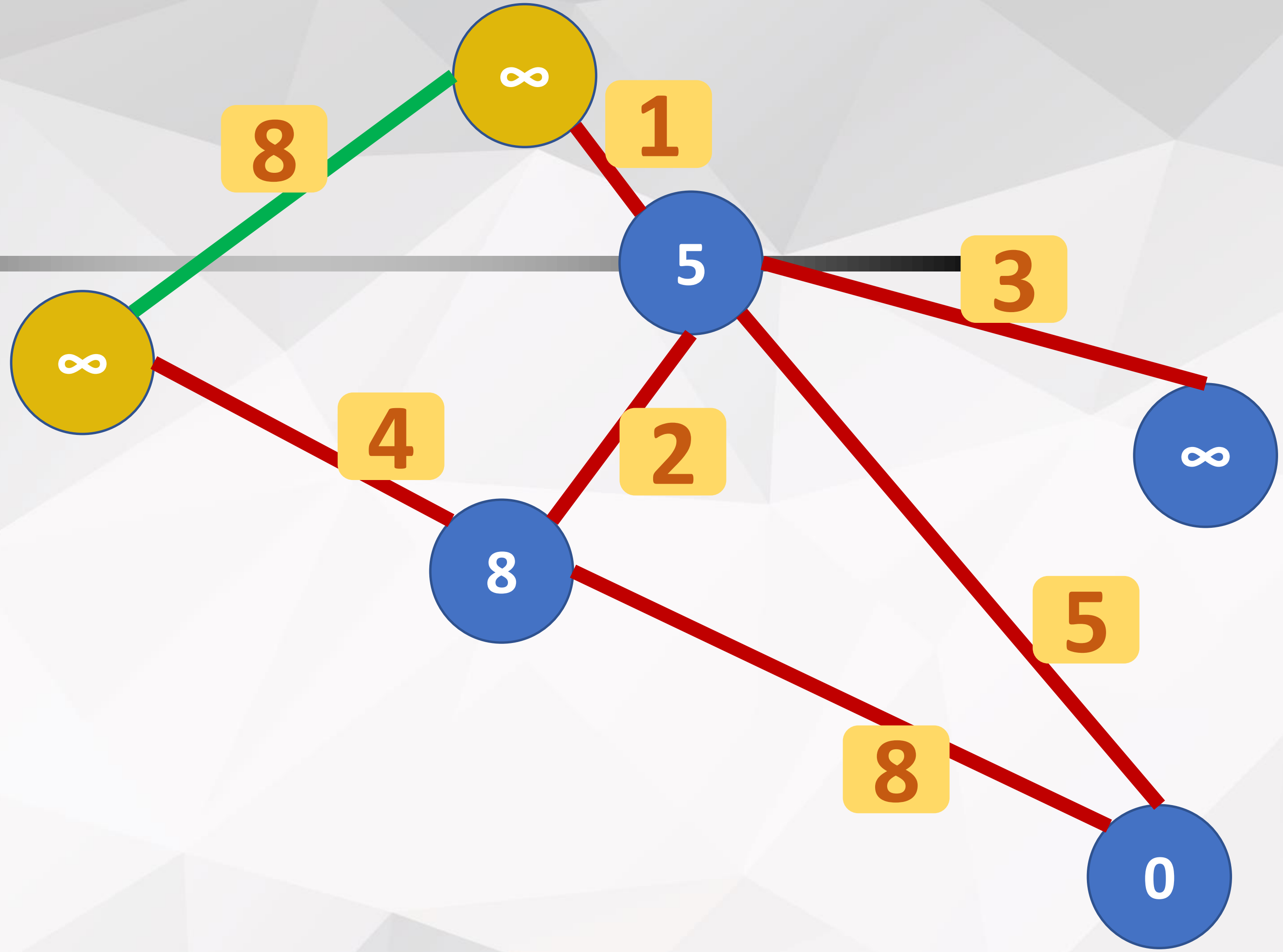

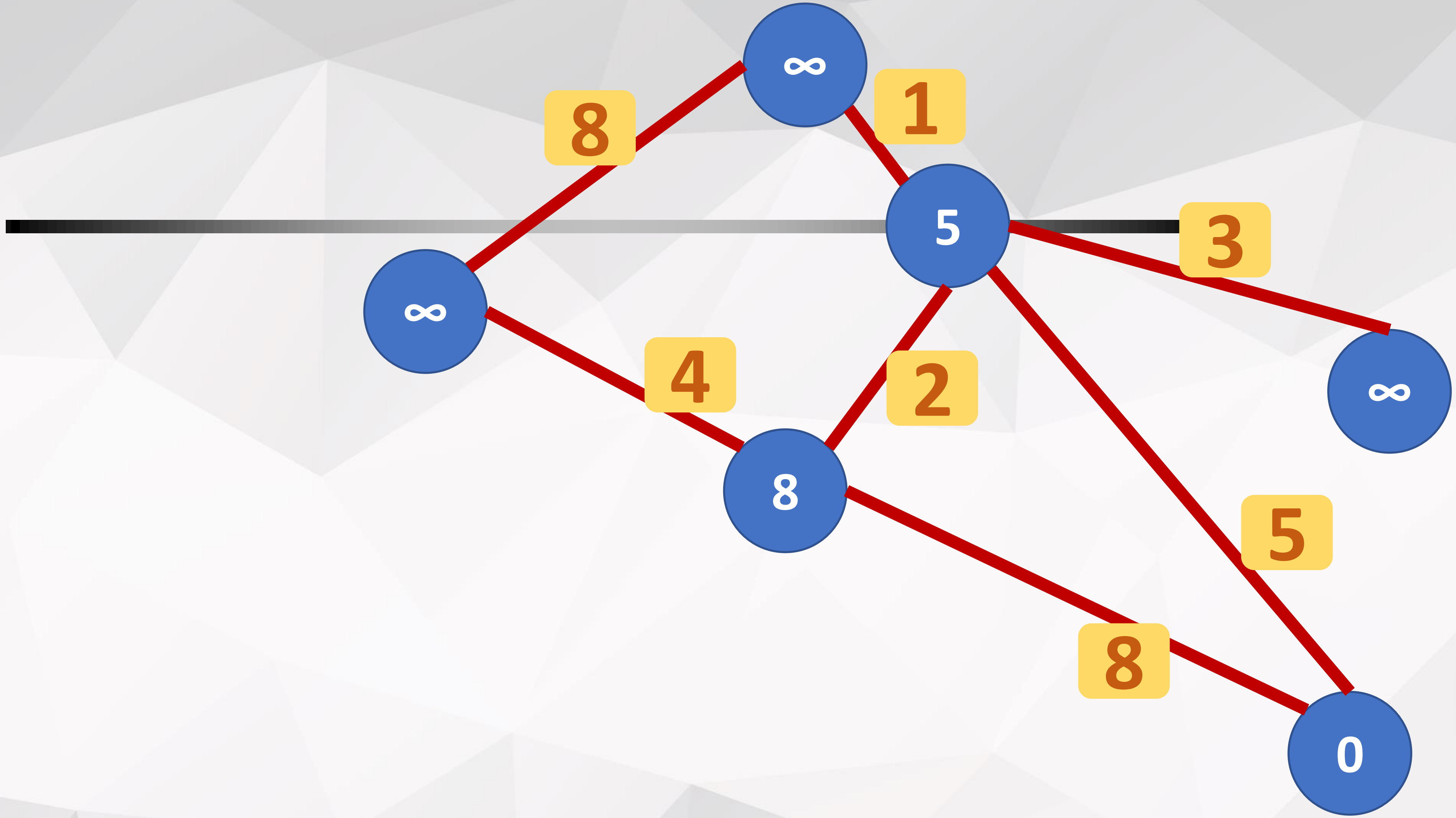

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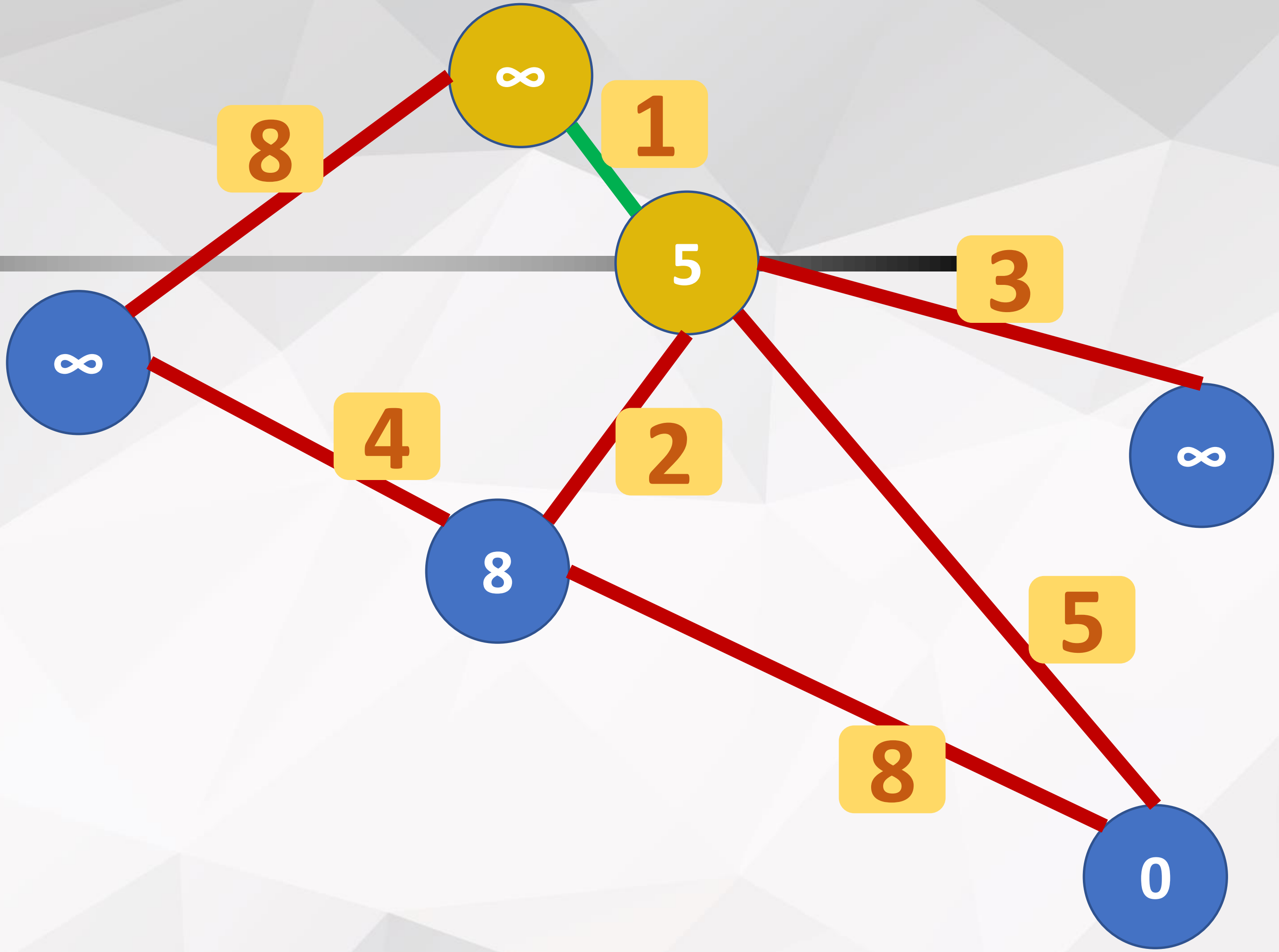
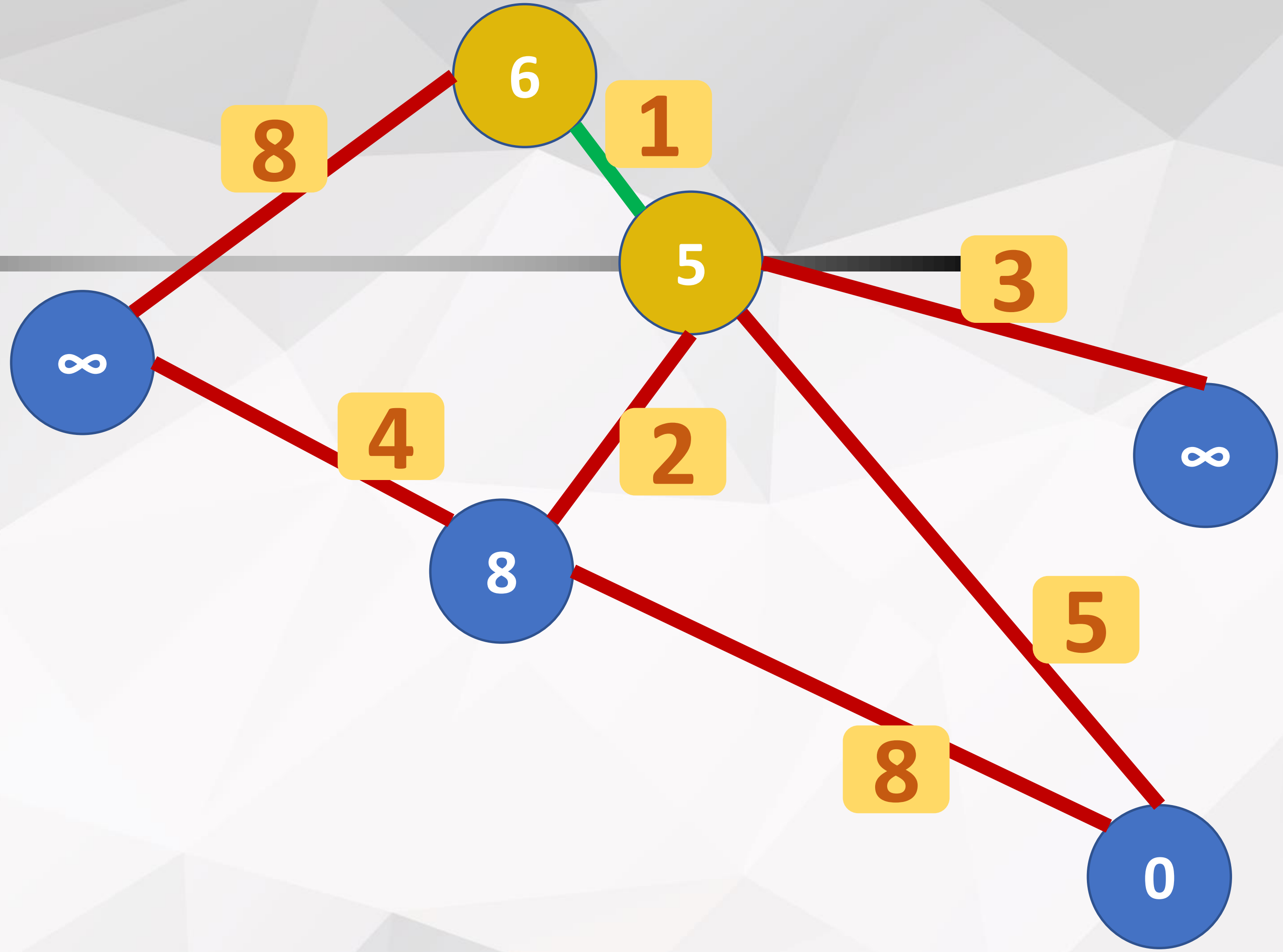
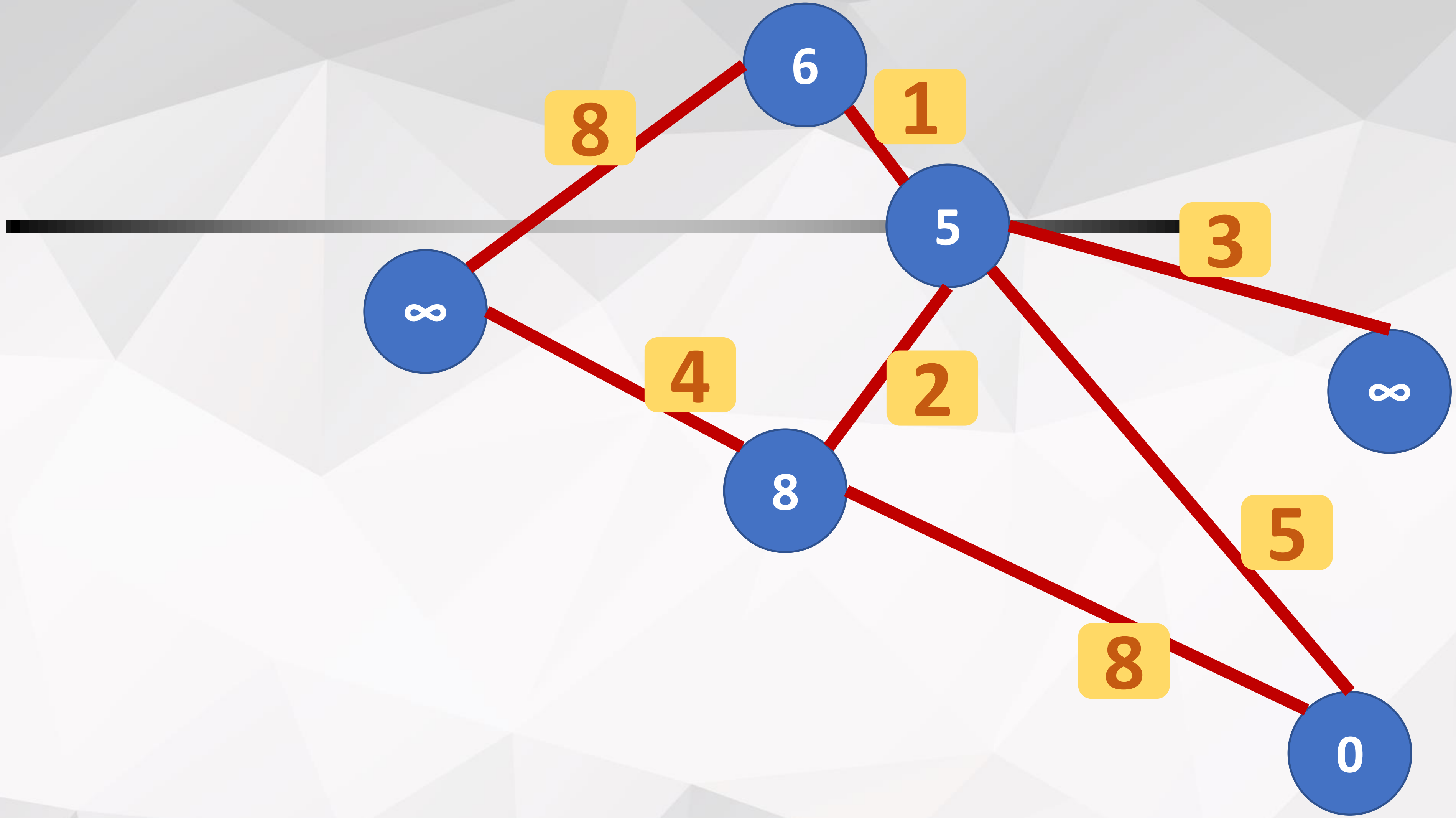


Competitive Programming

Relax!

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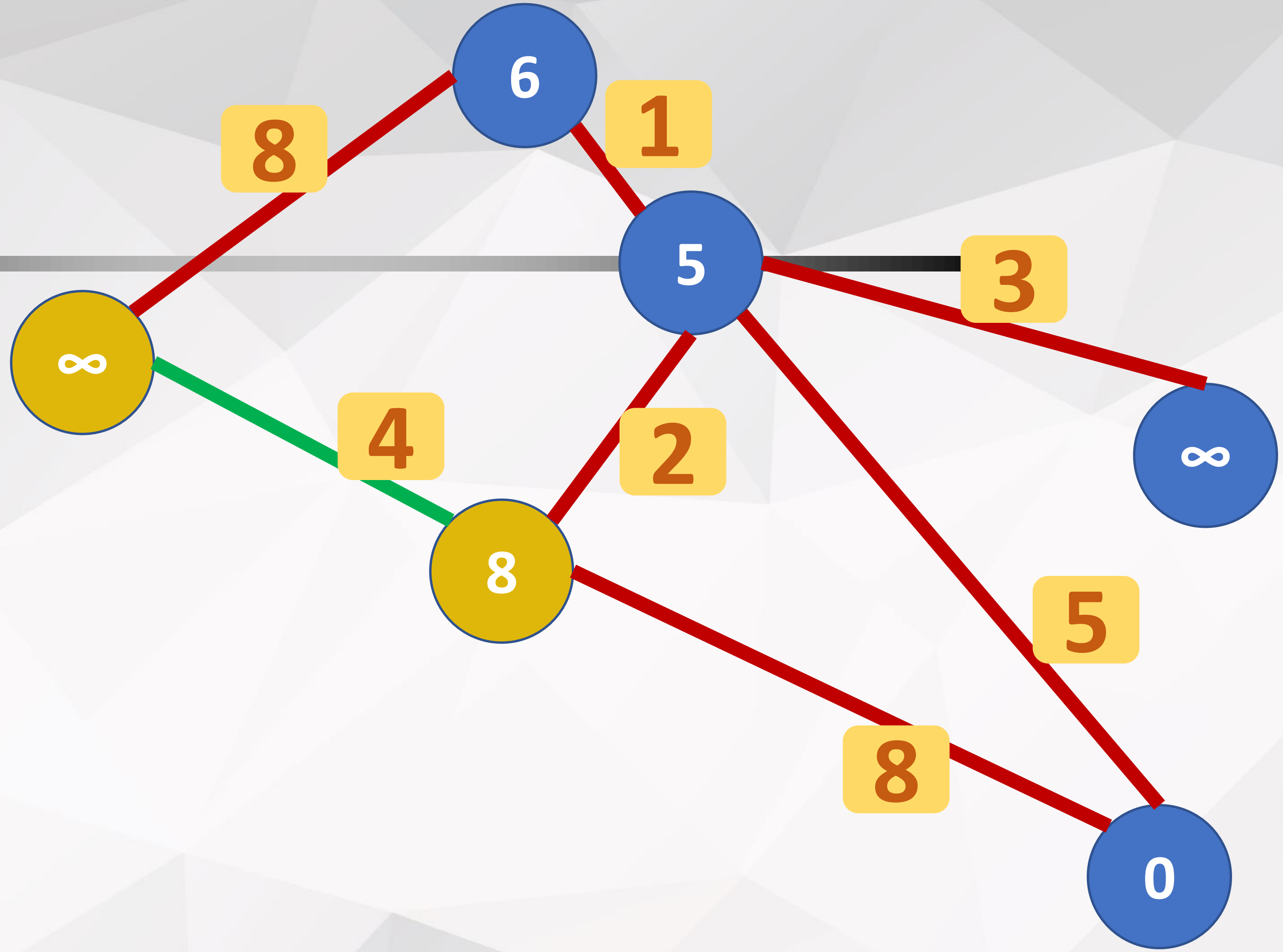

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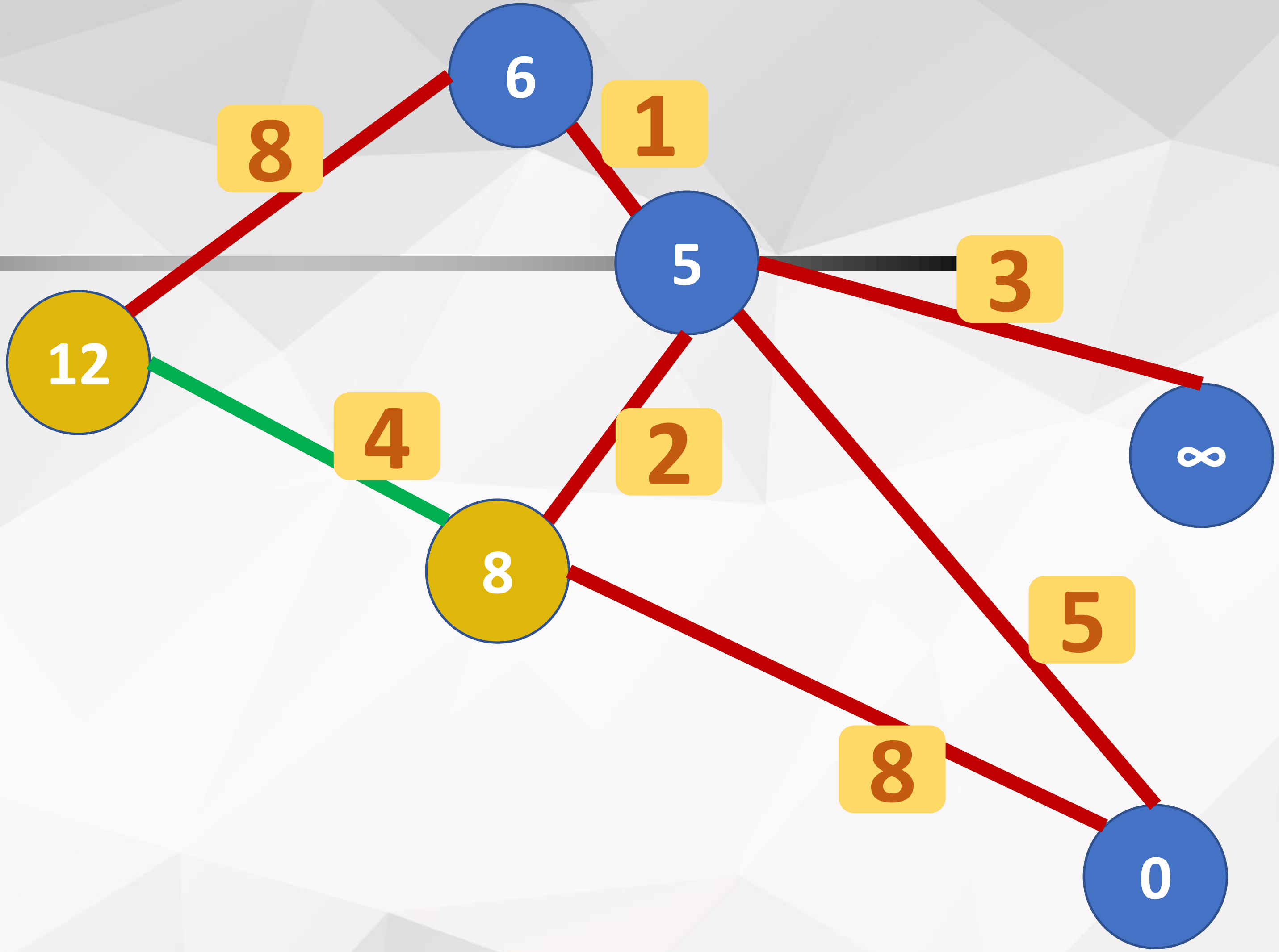

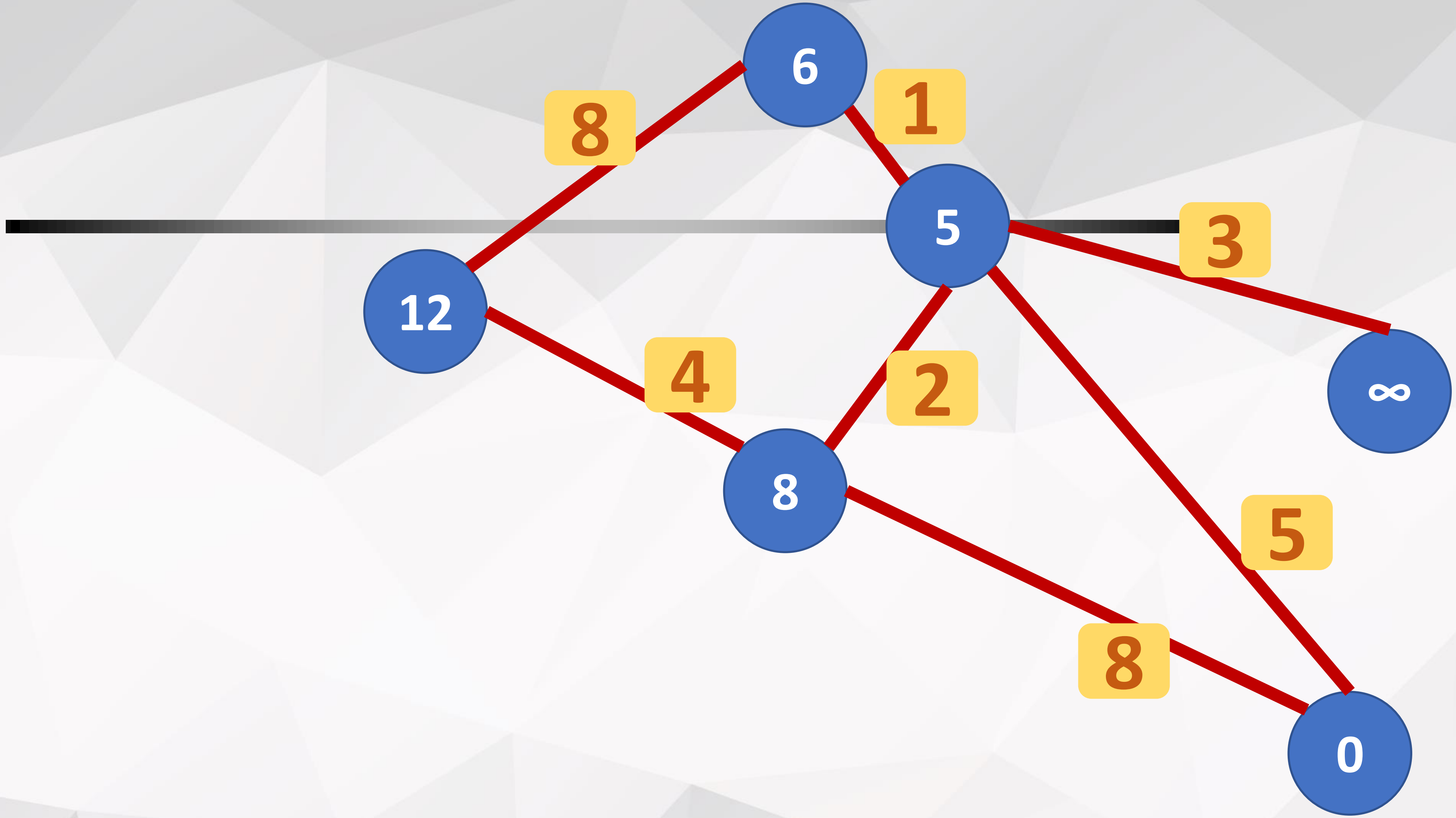
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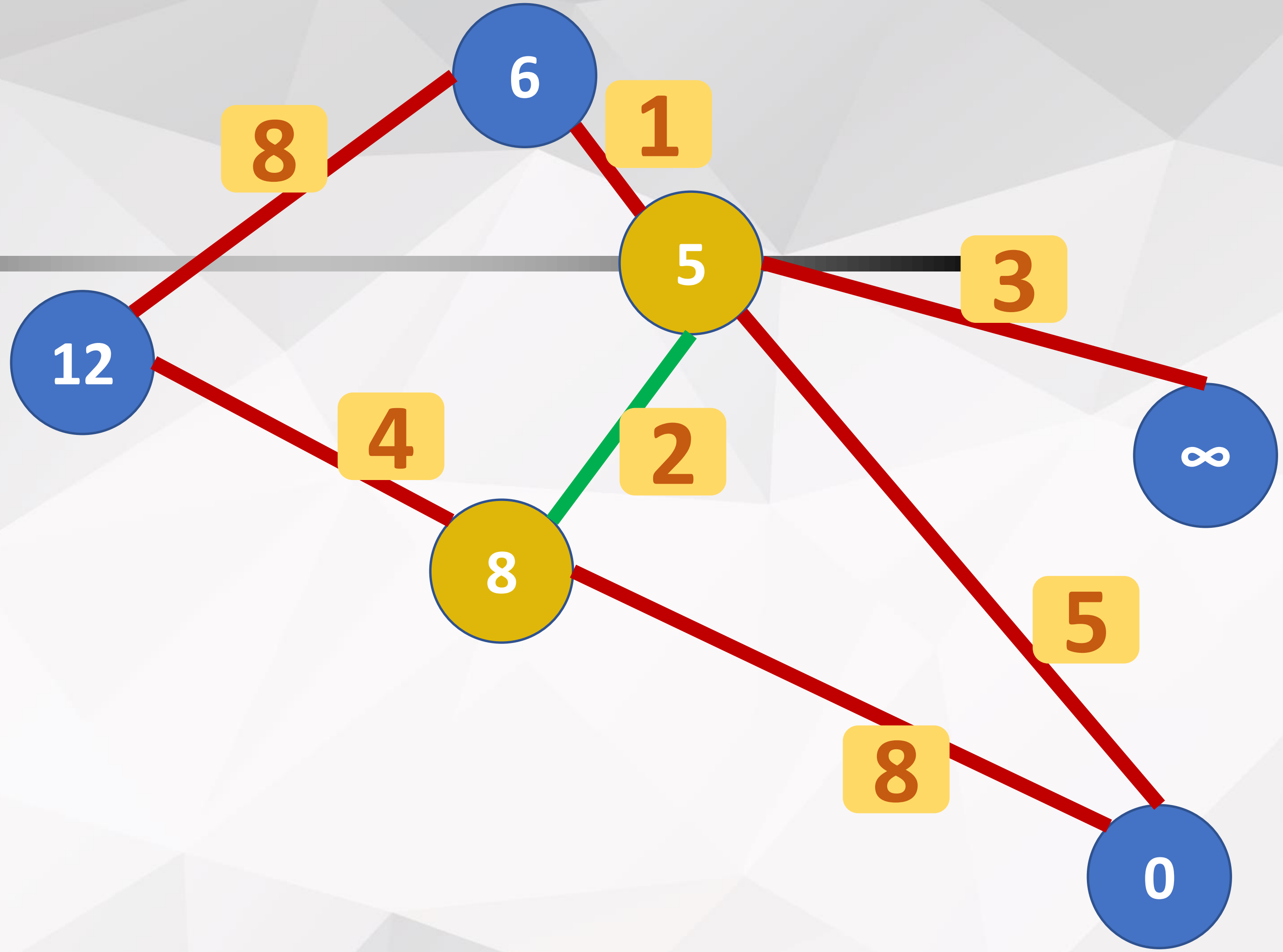

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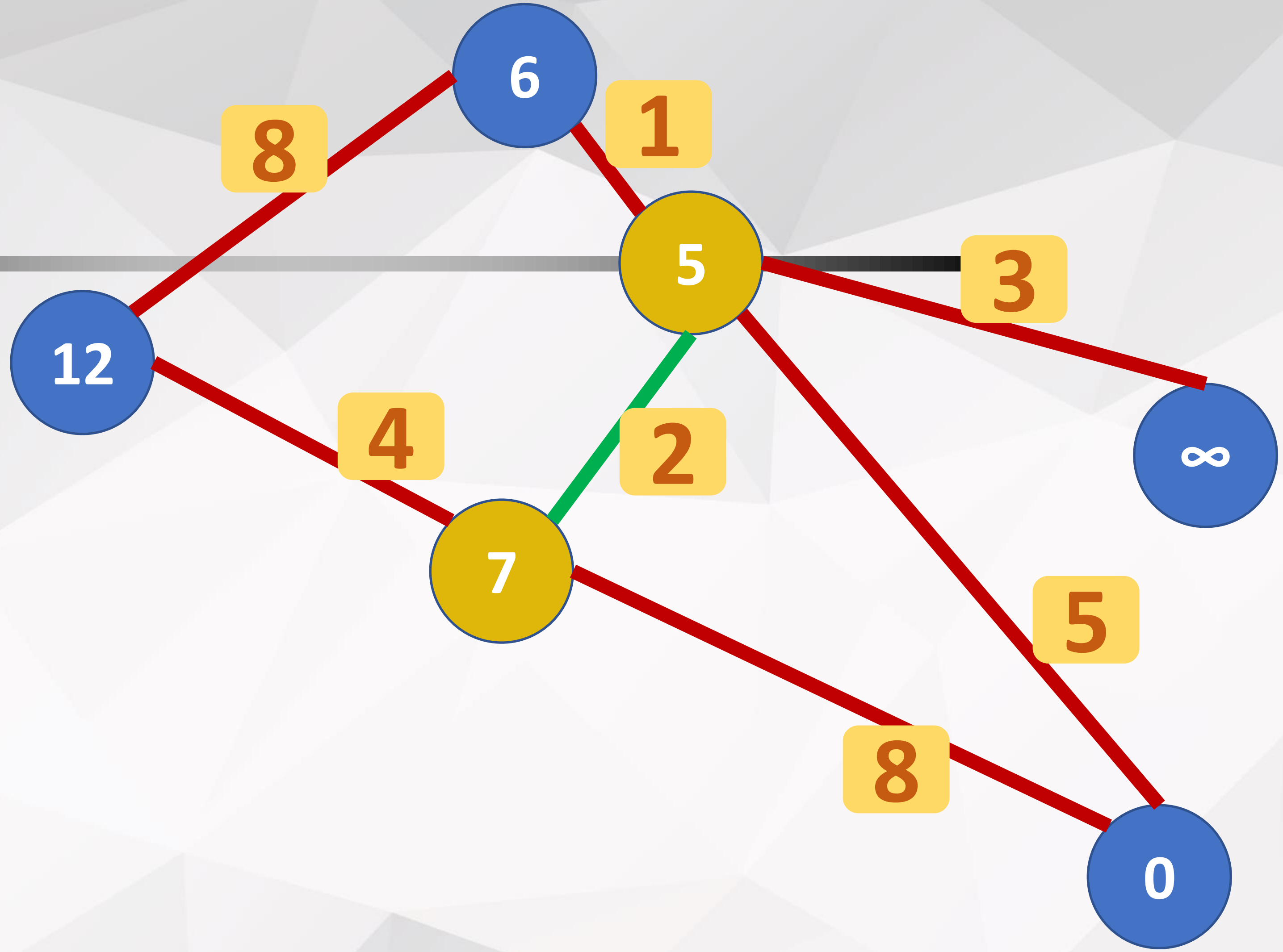

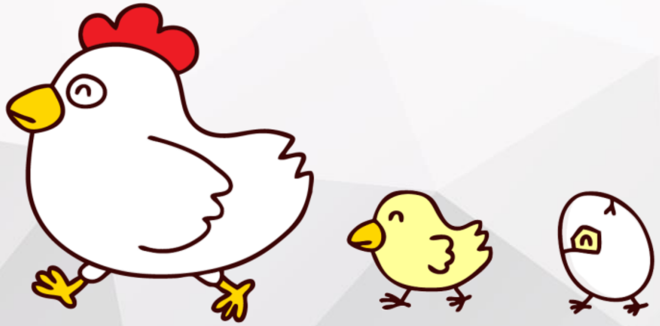

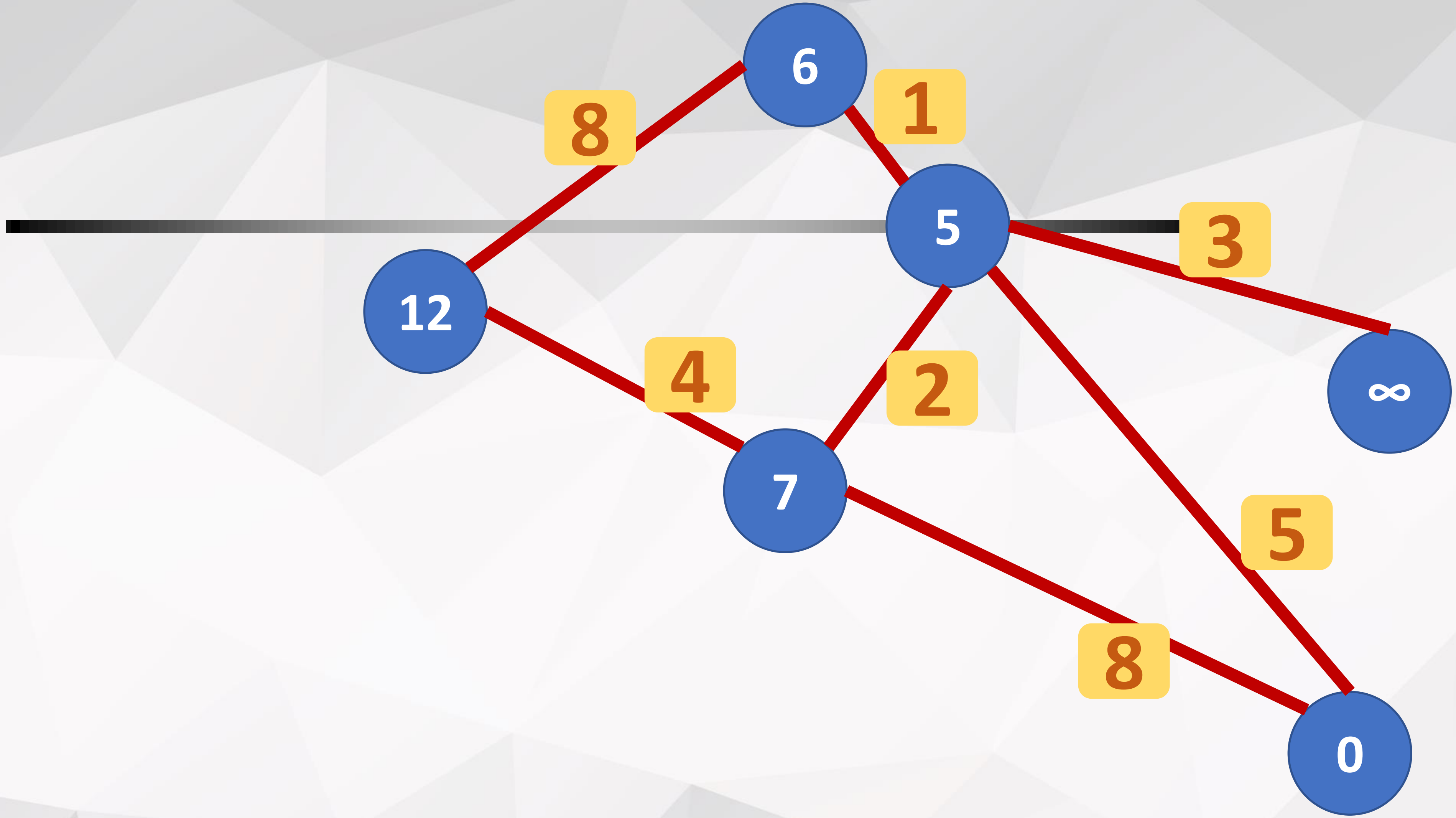

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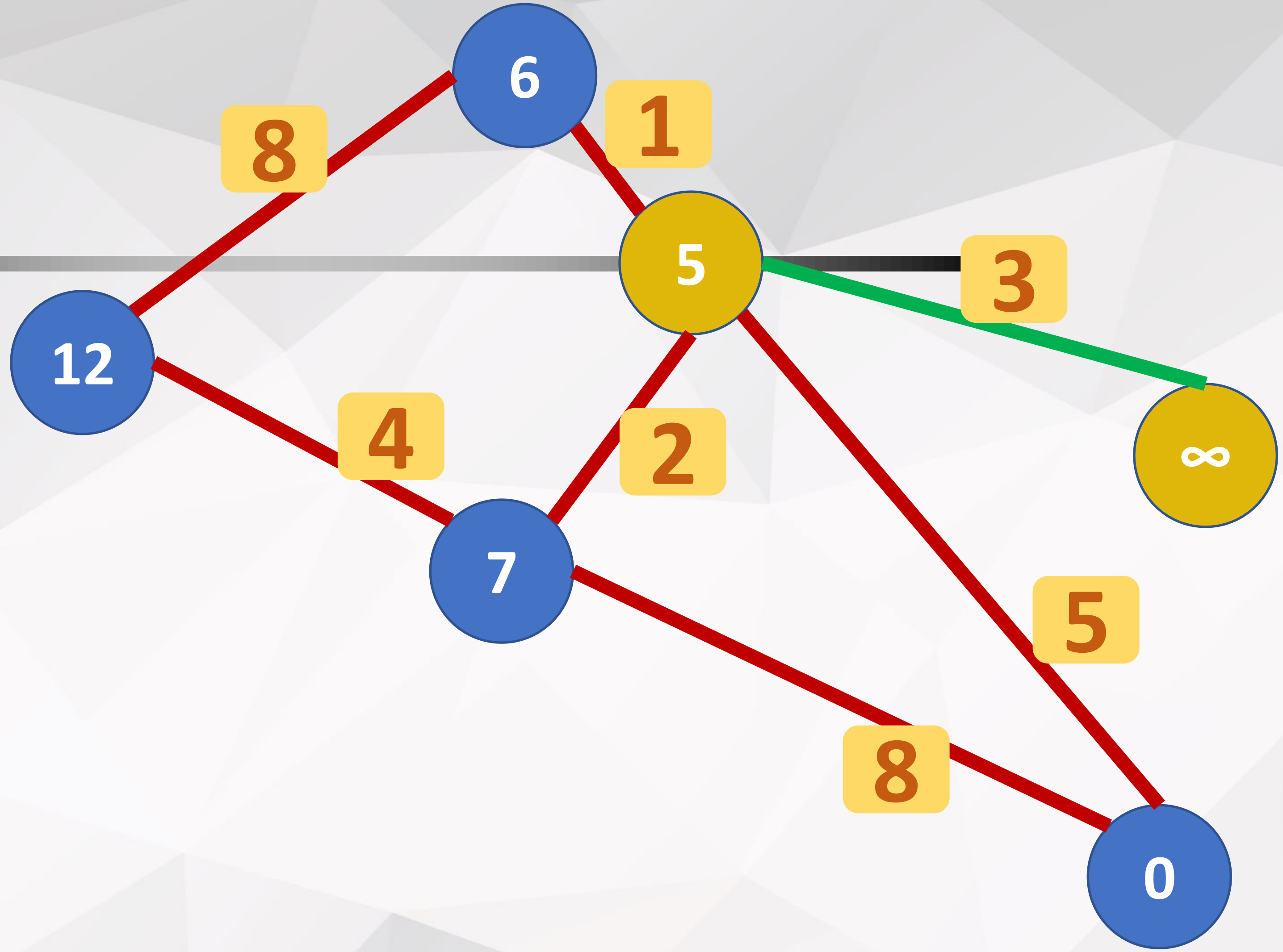

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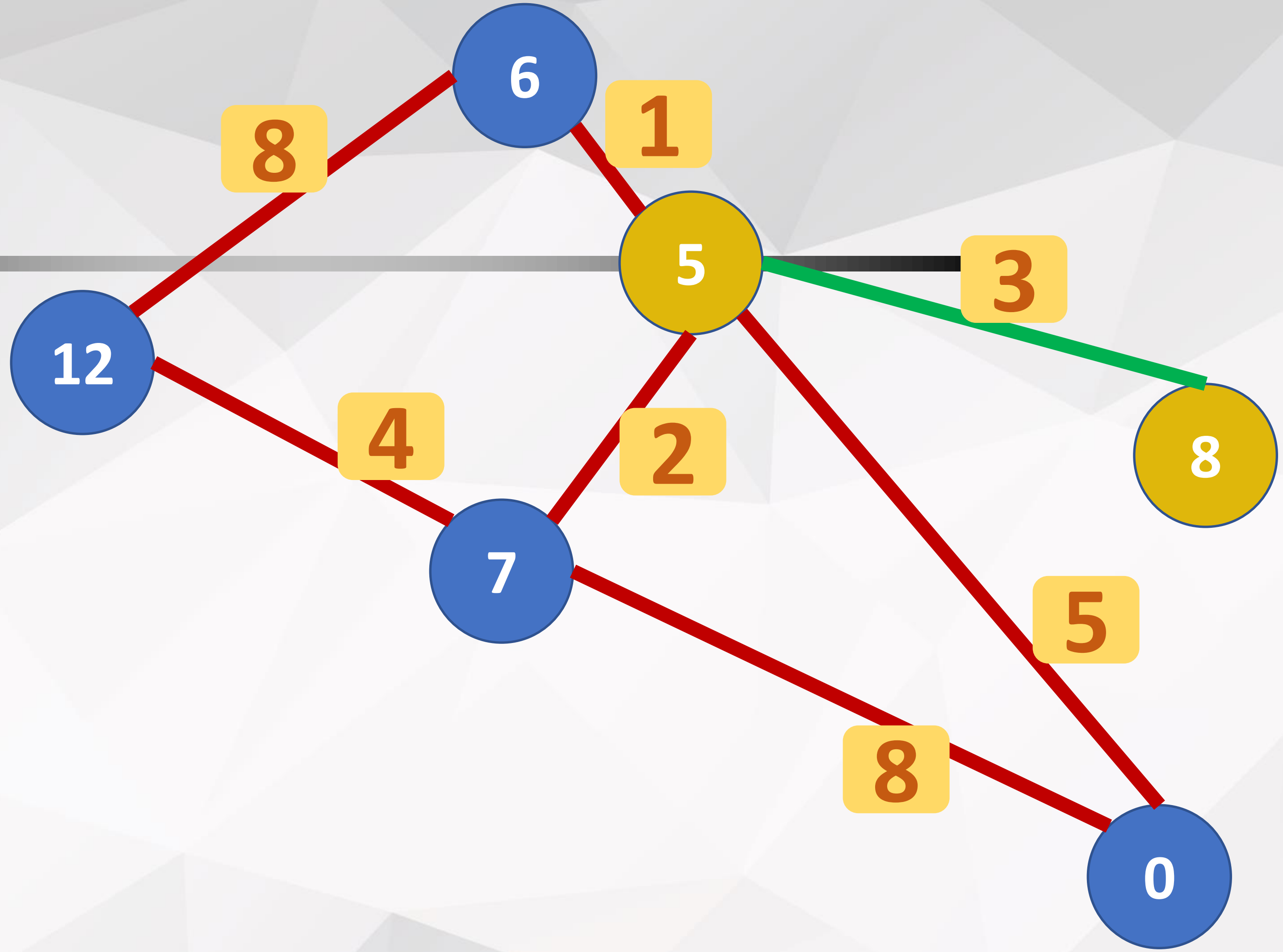

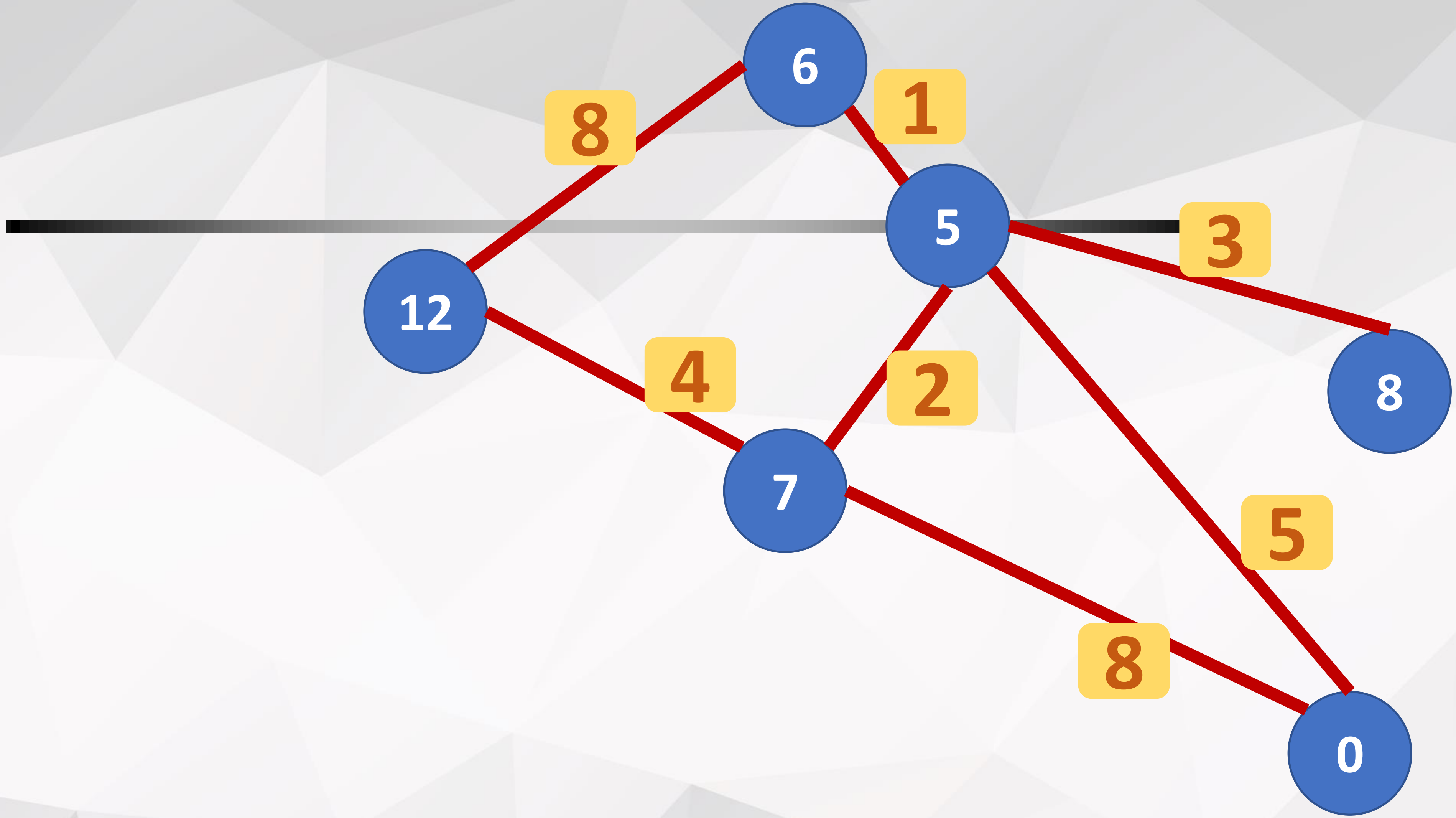
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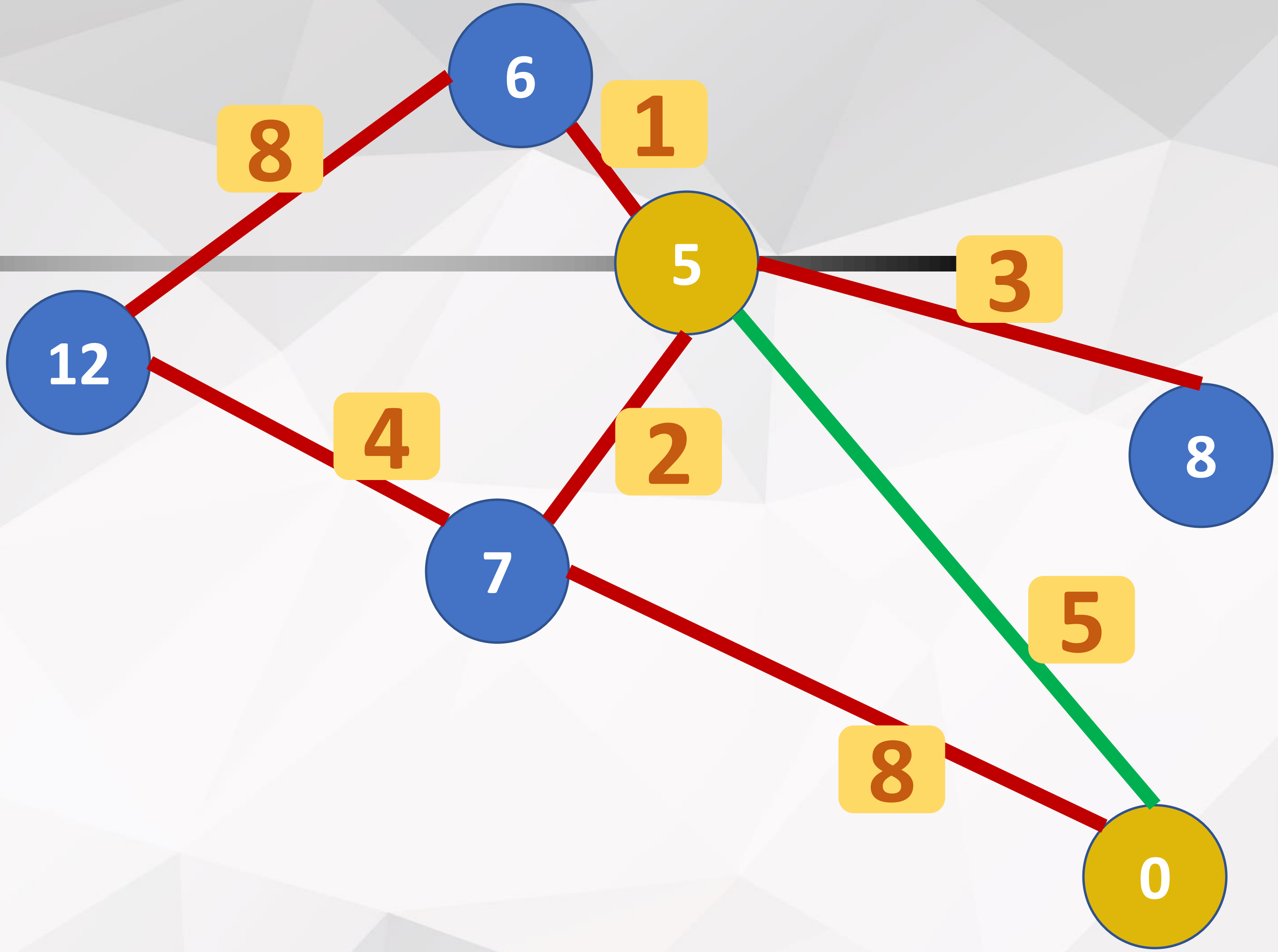

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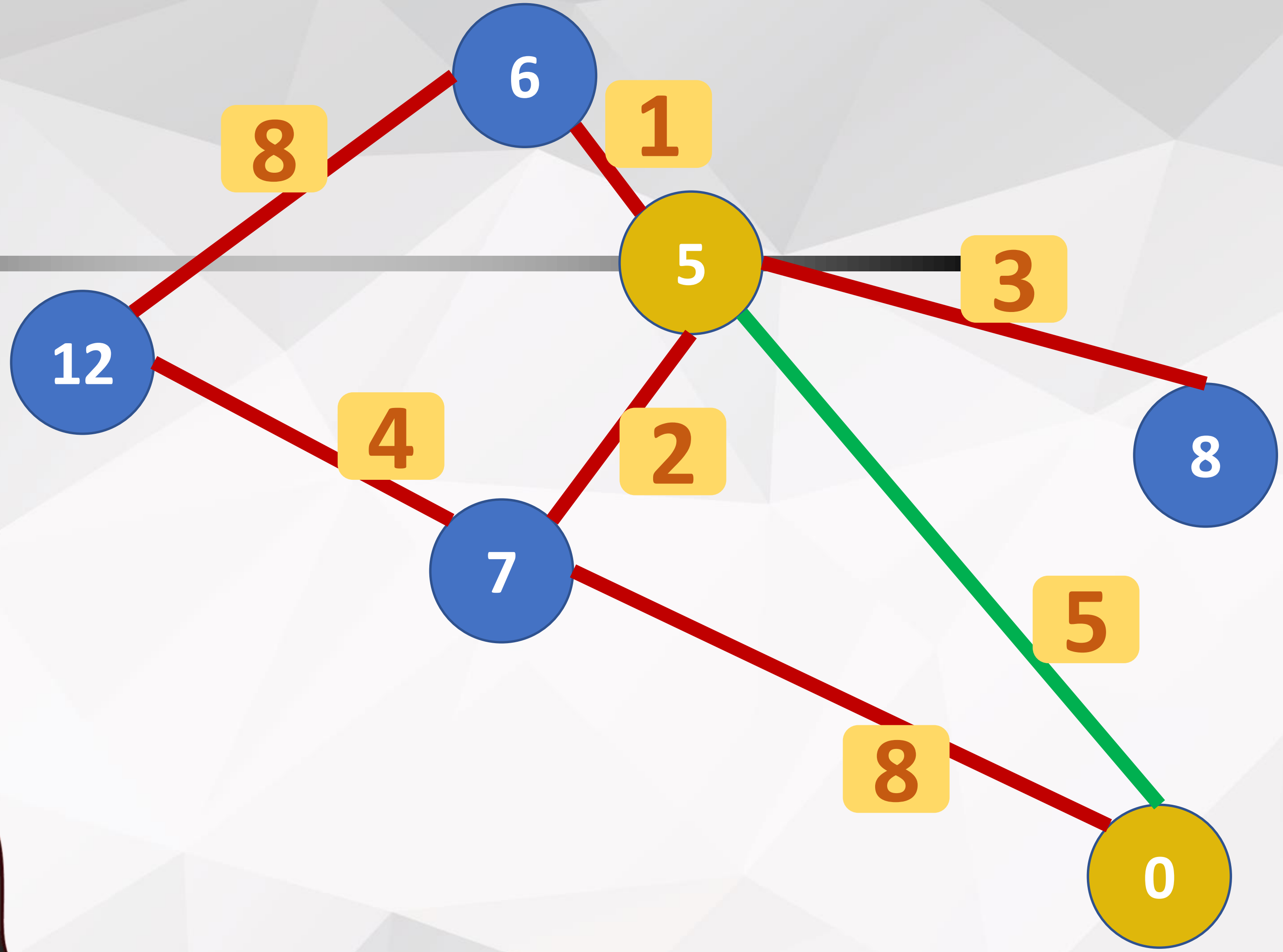

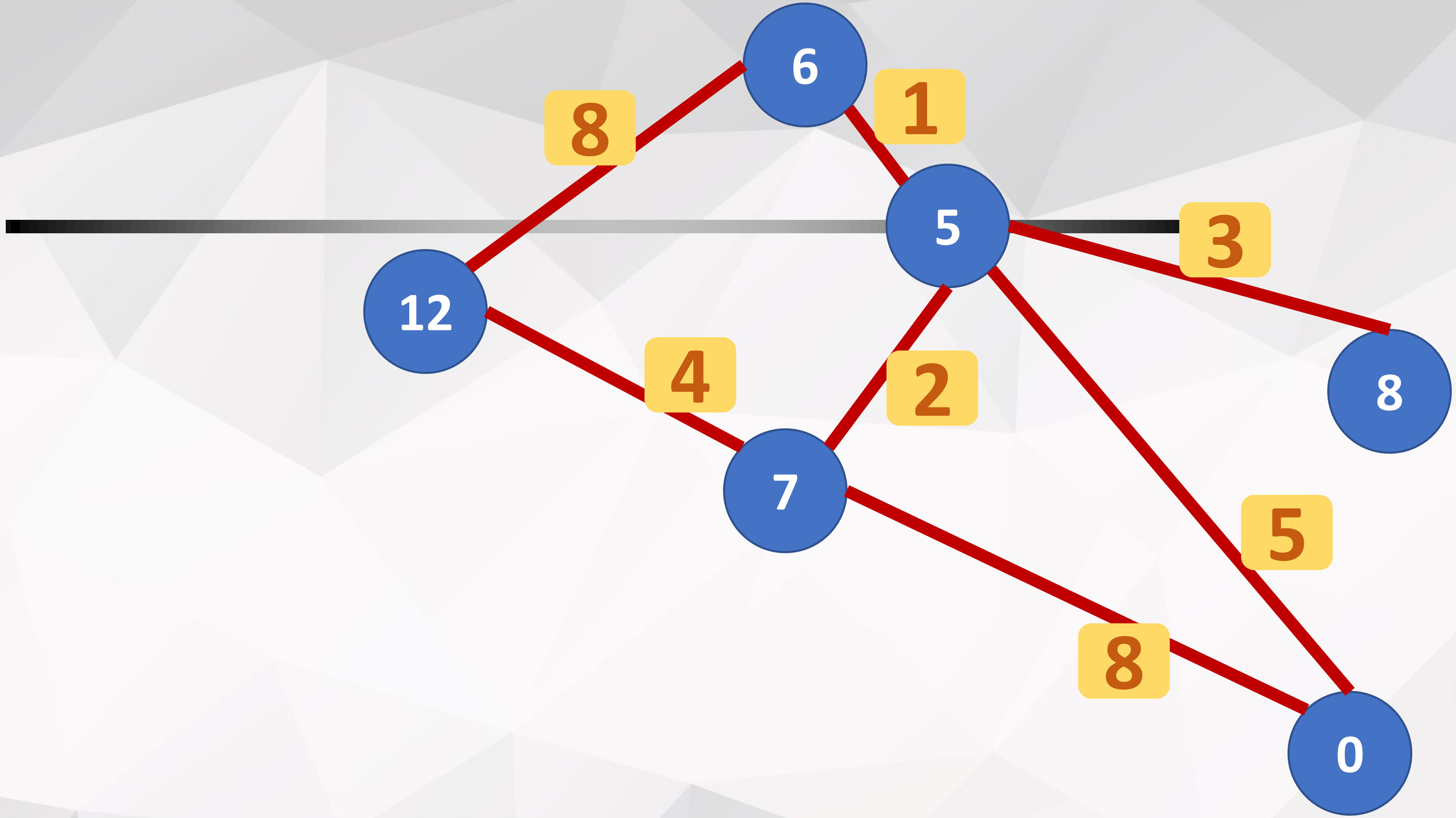
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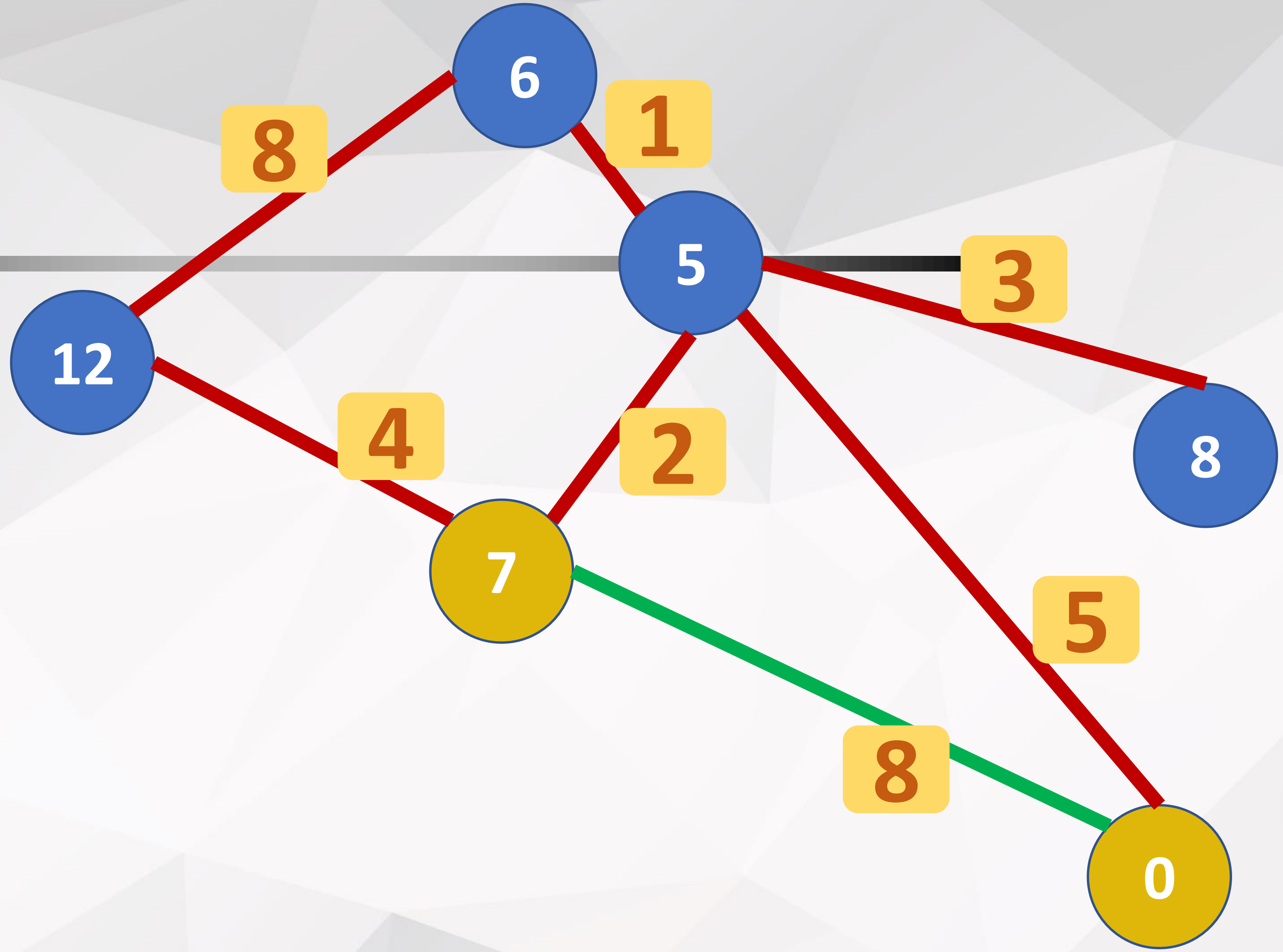

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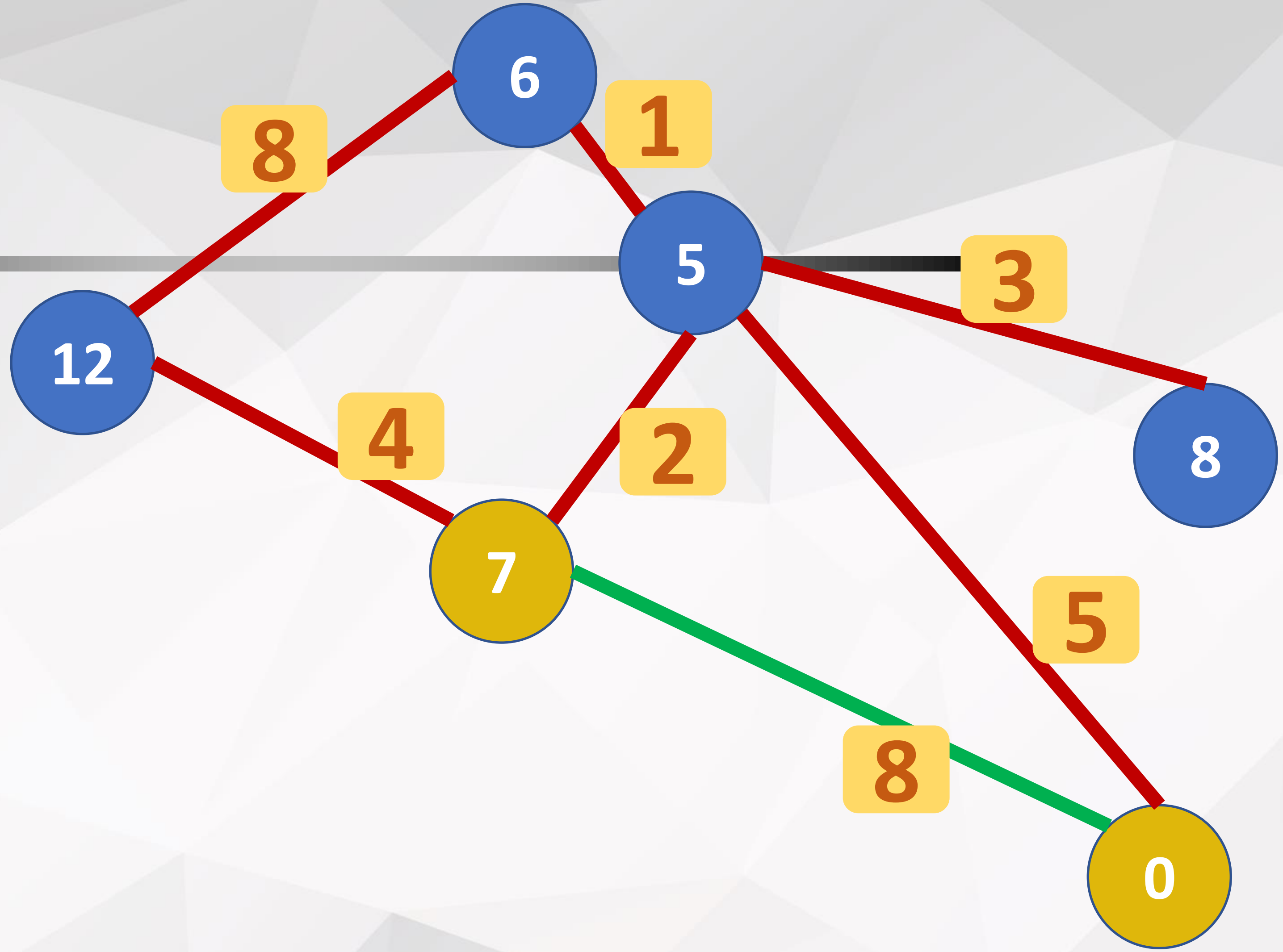

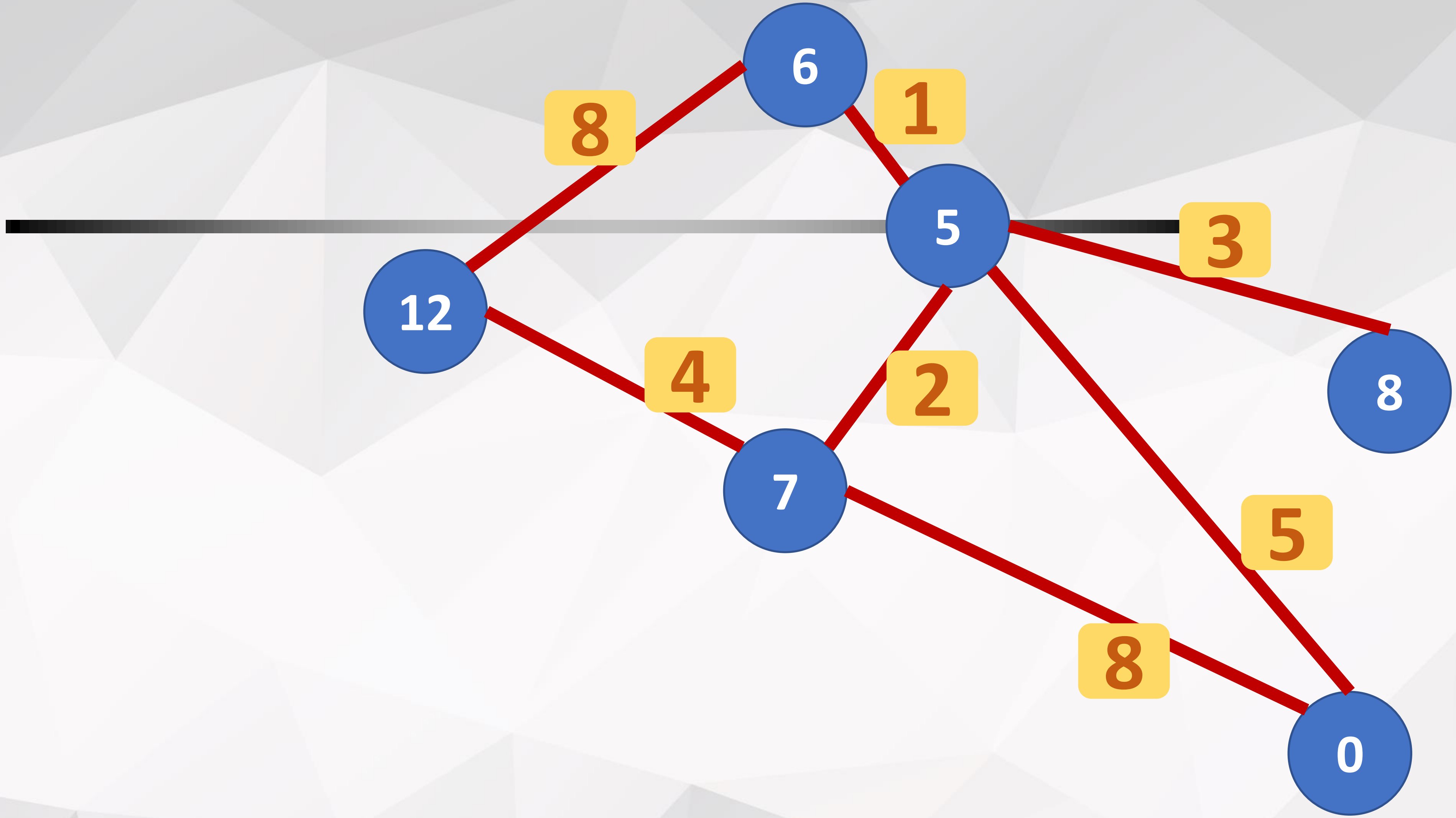
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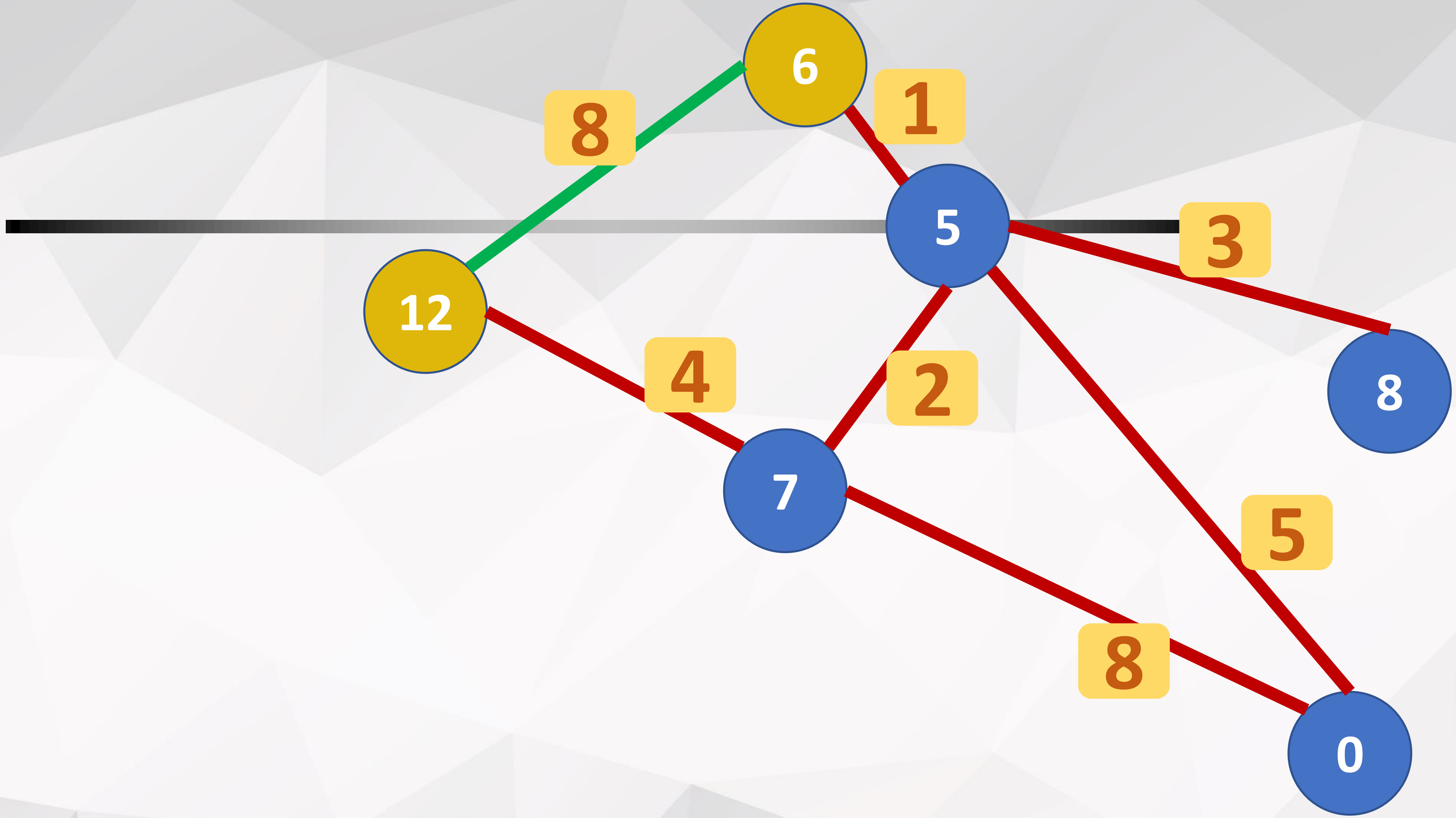
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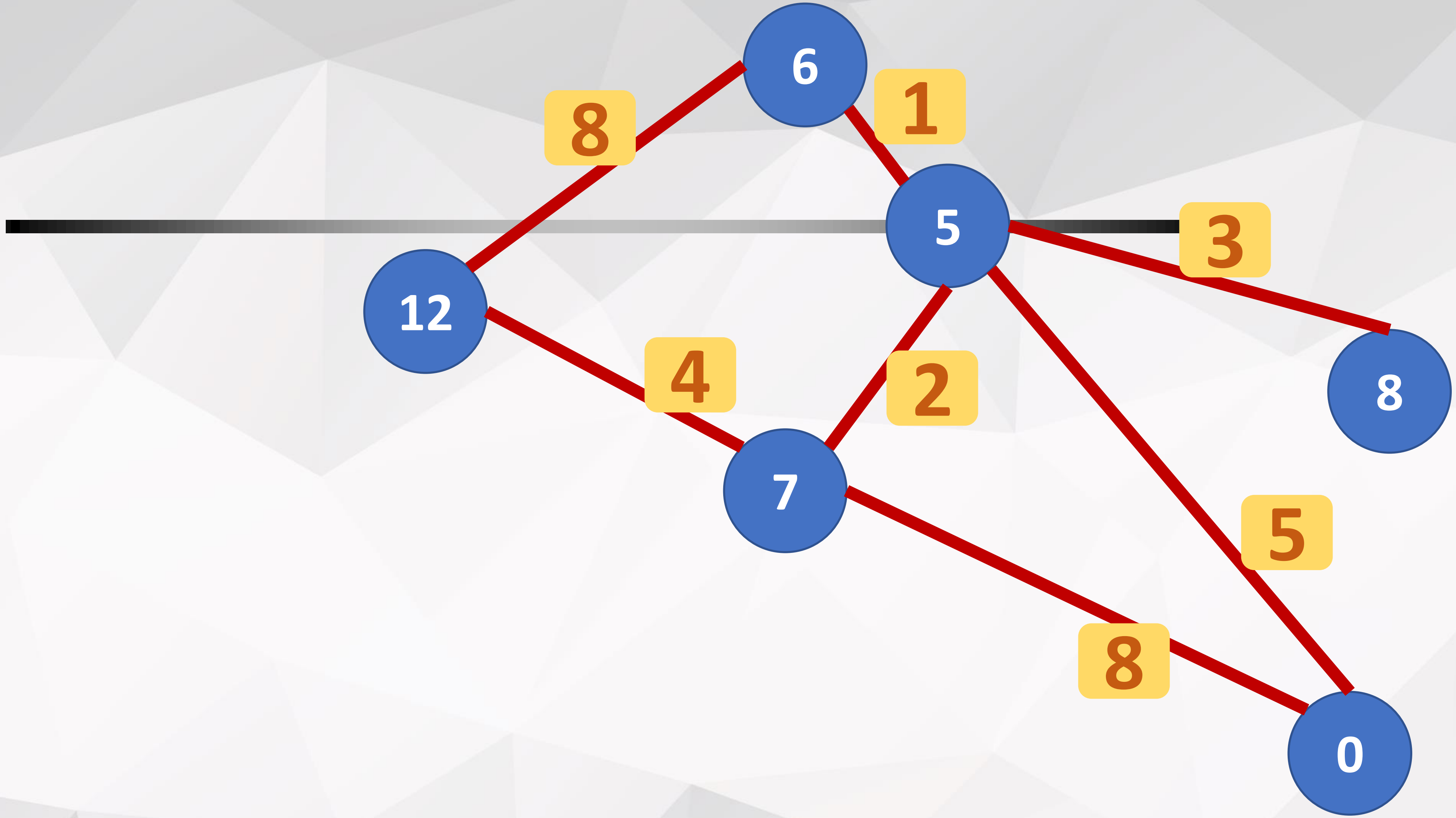
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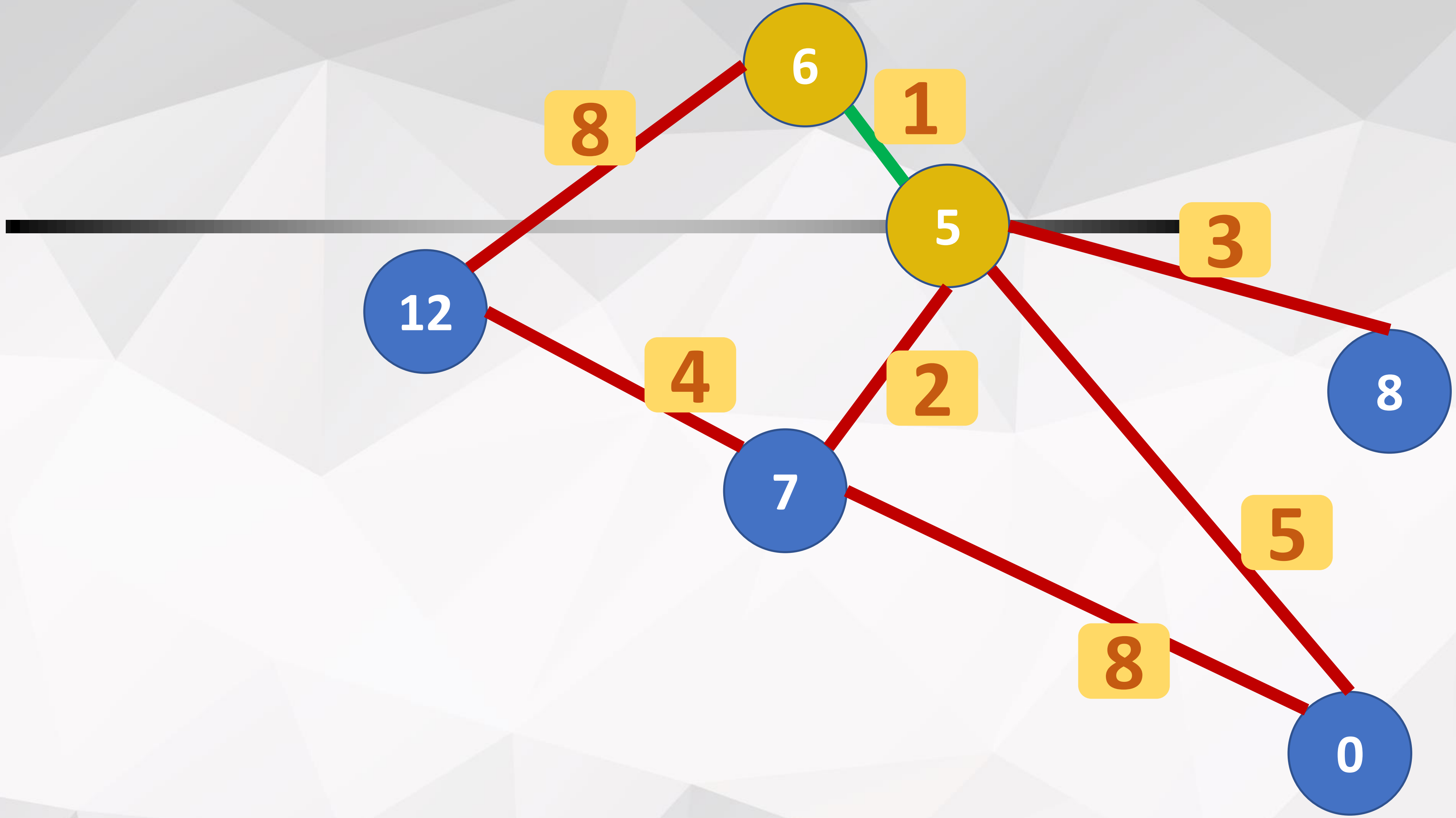
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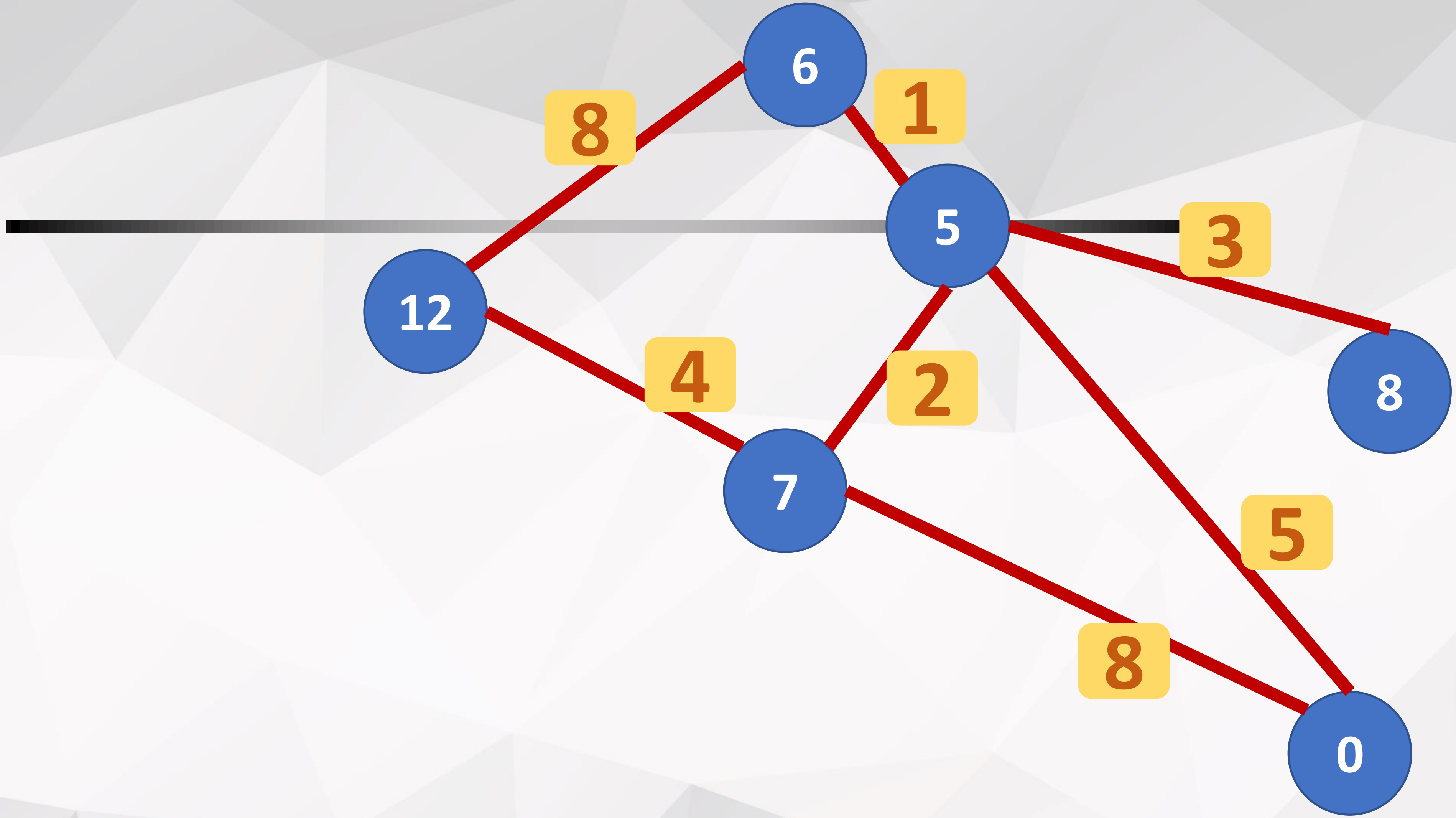
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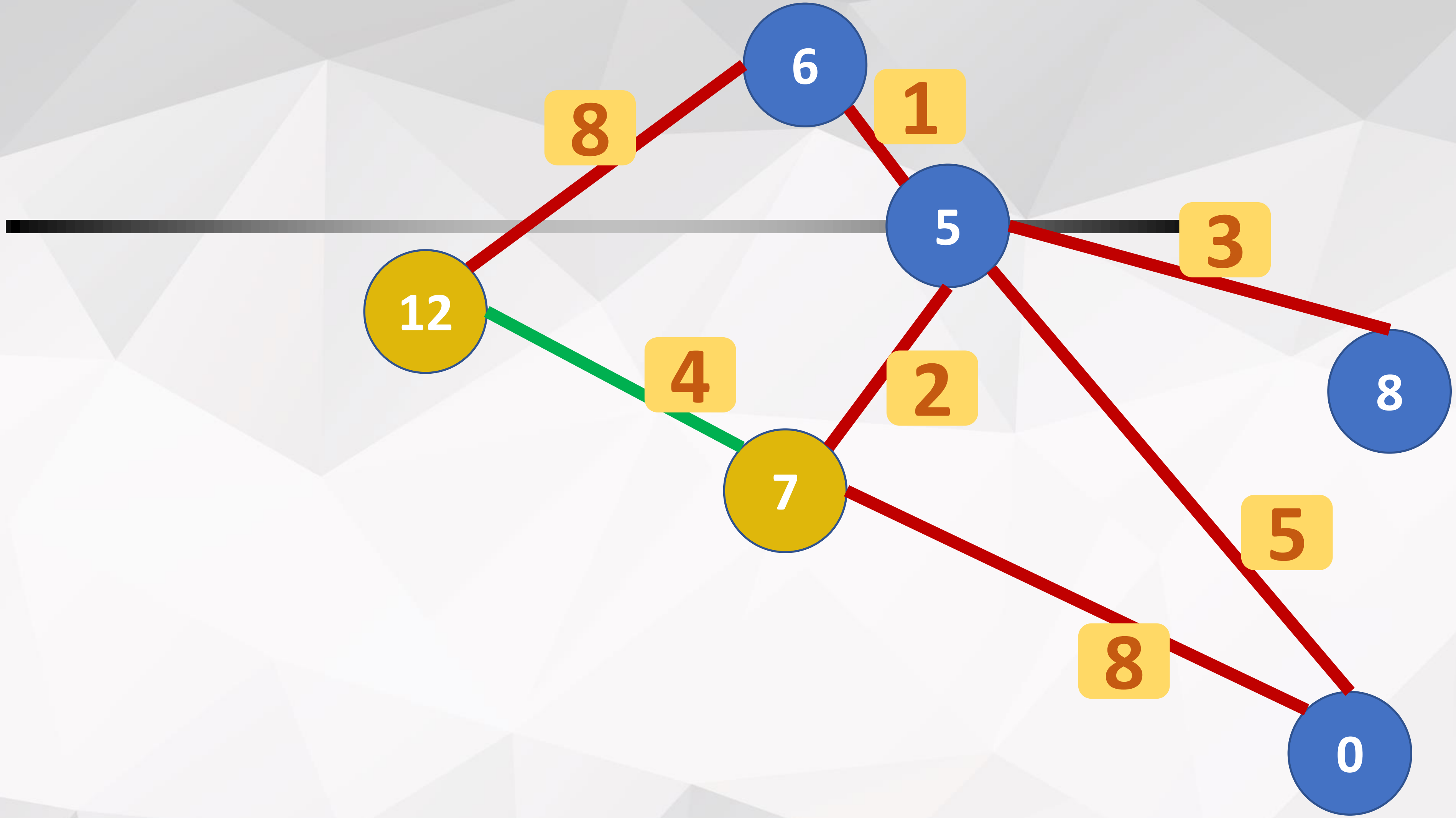
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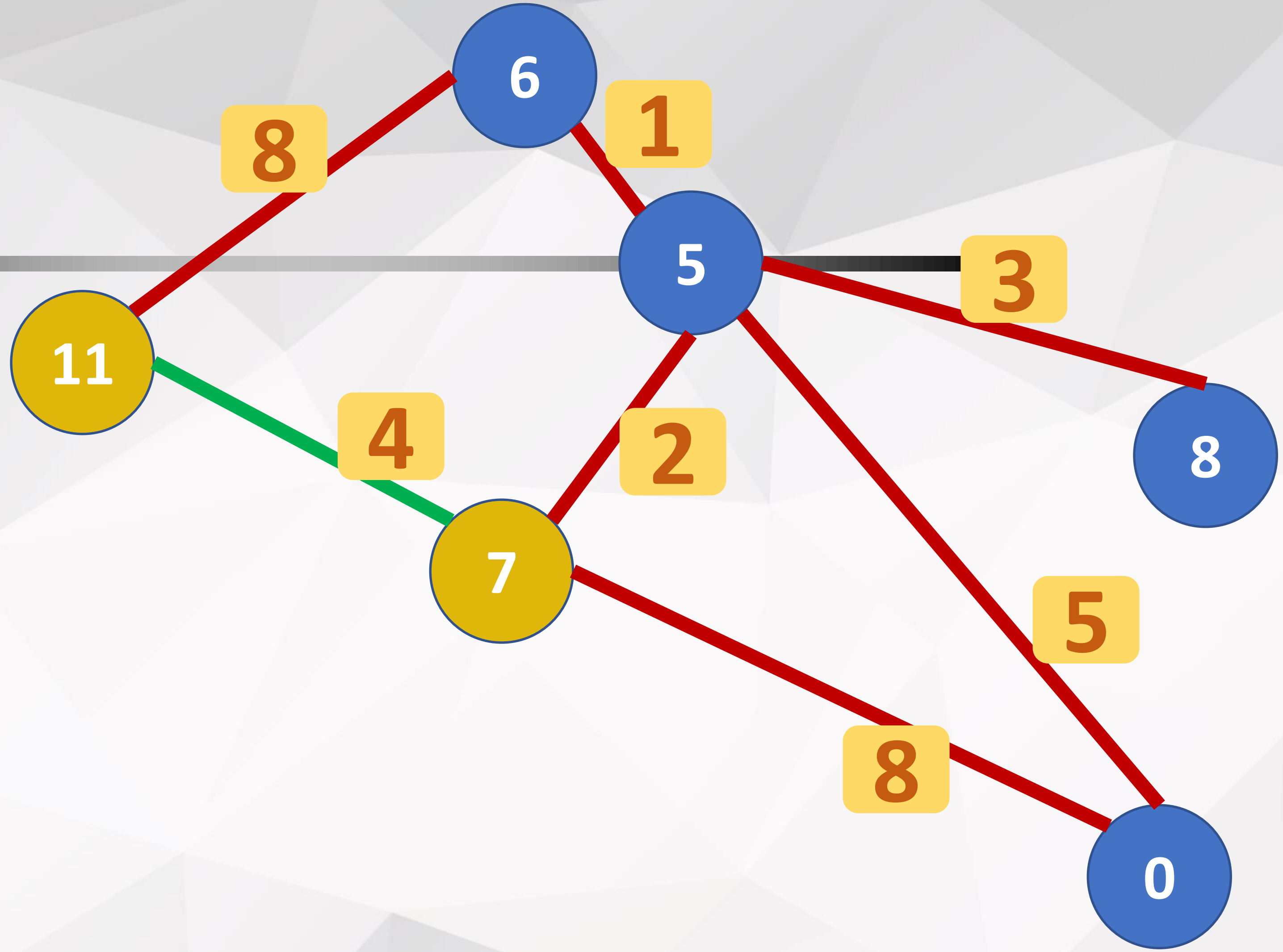

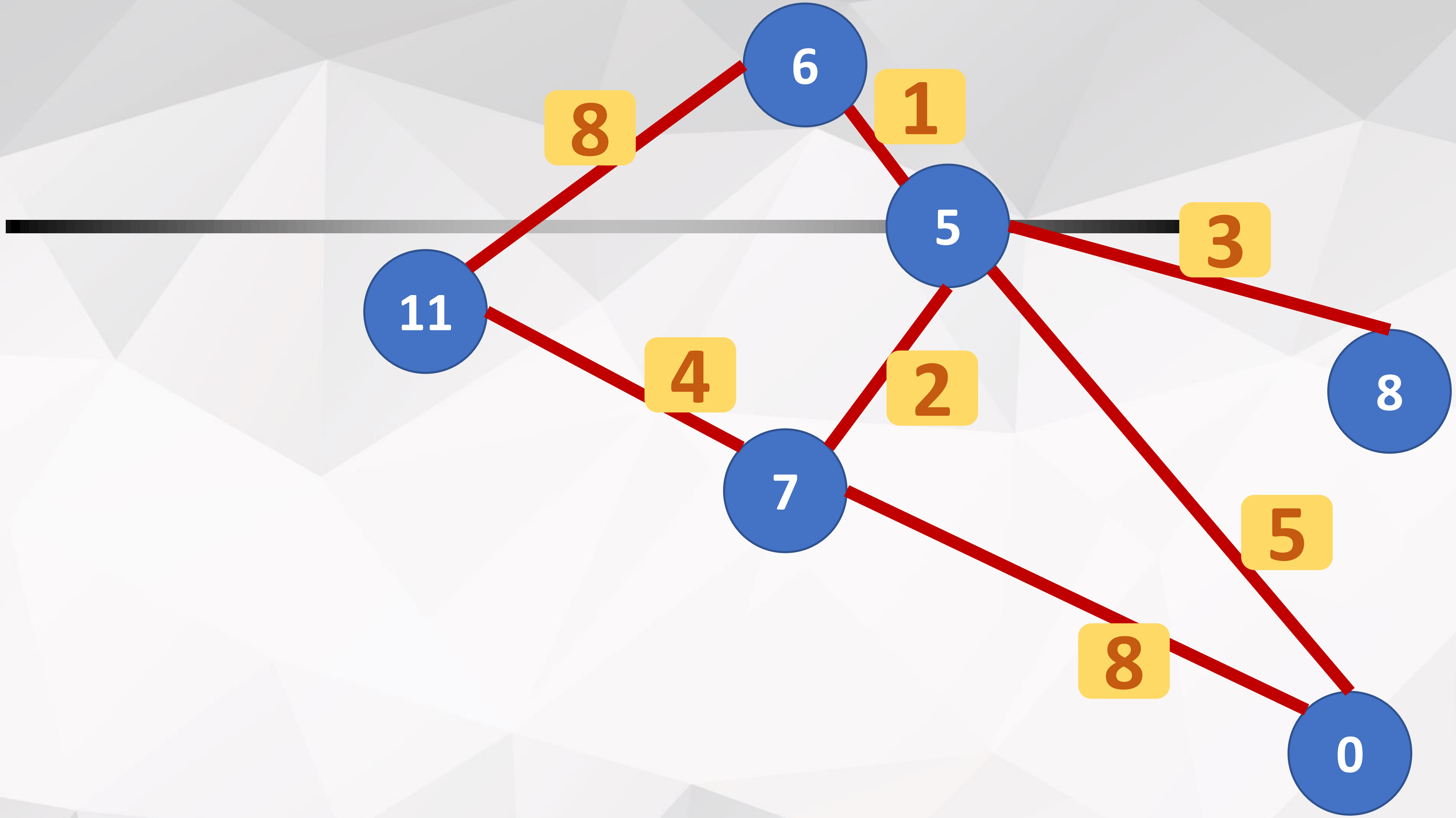
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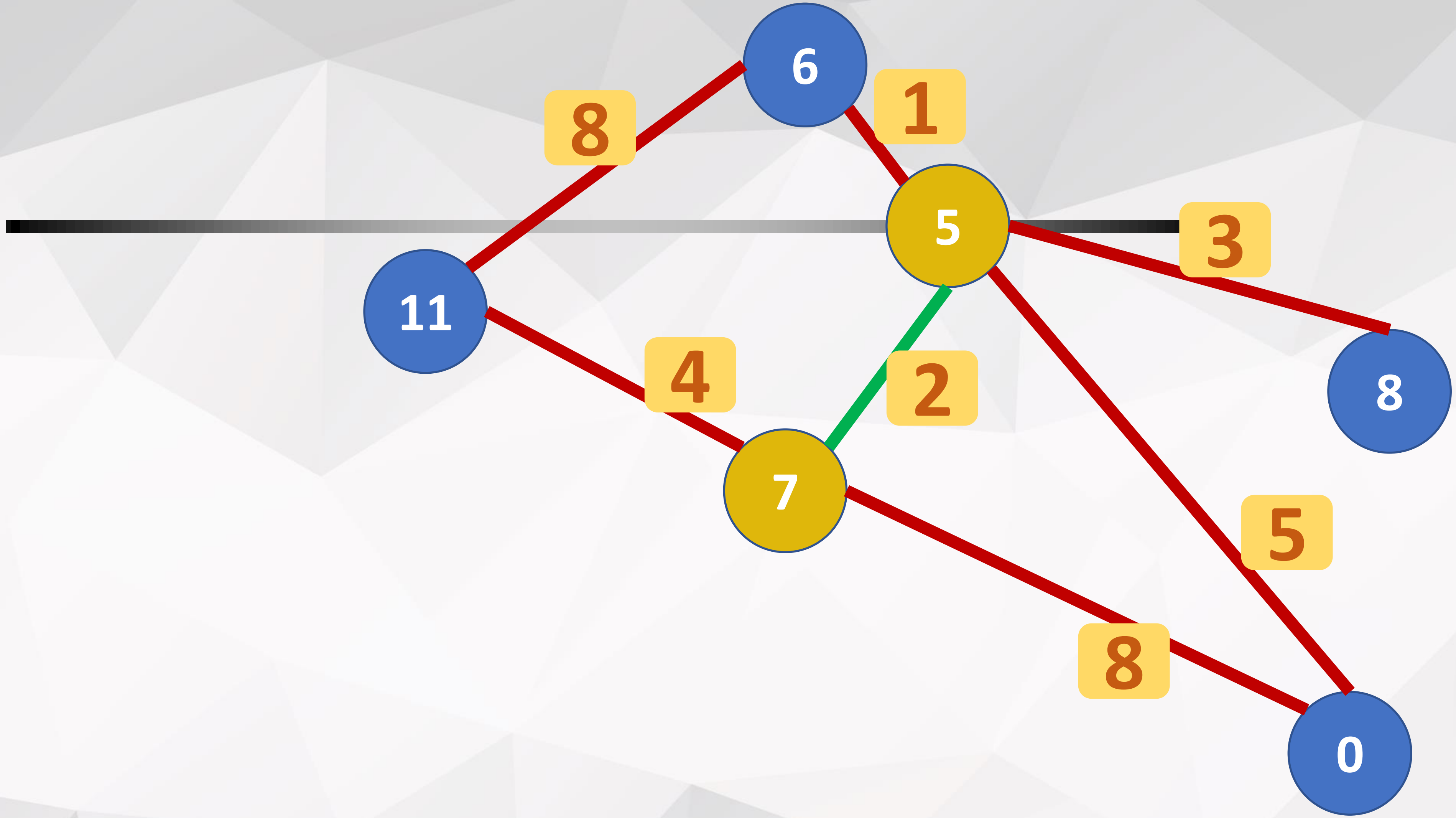

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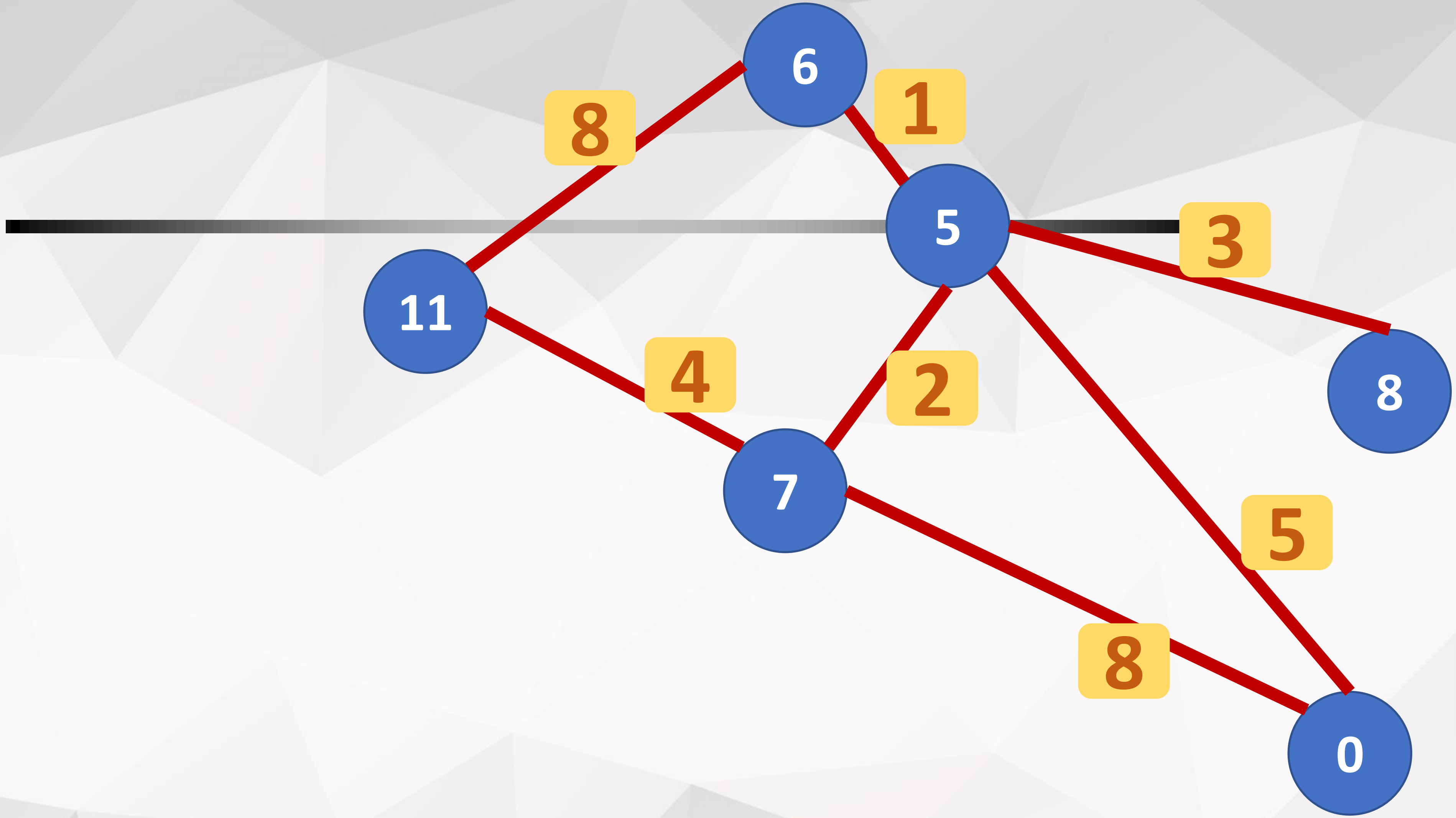
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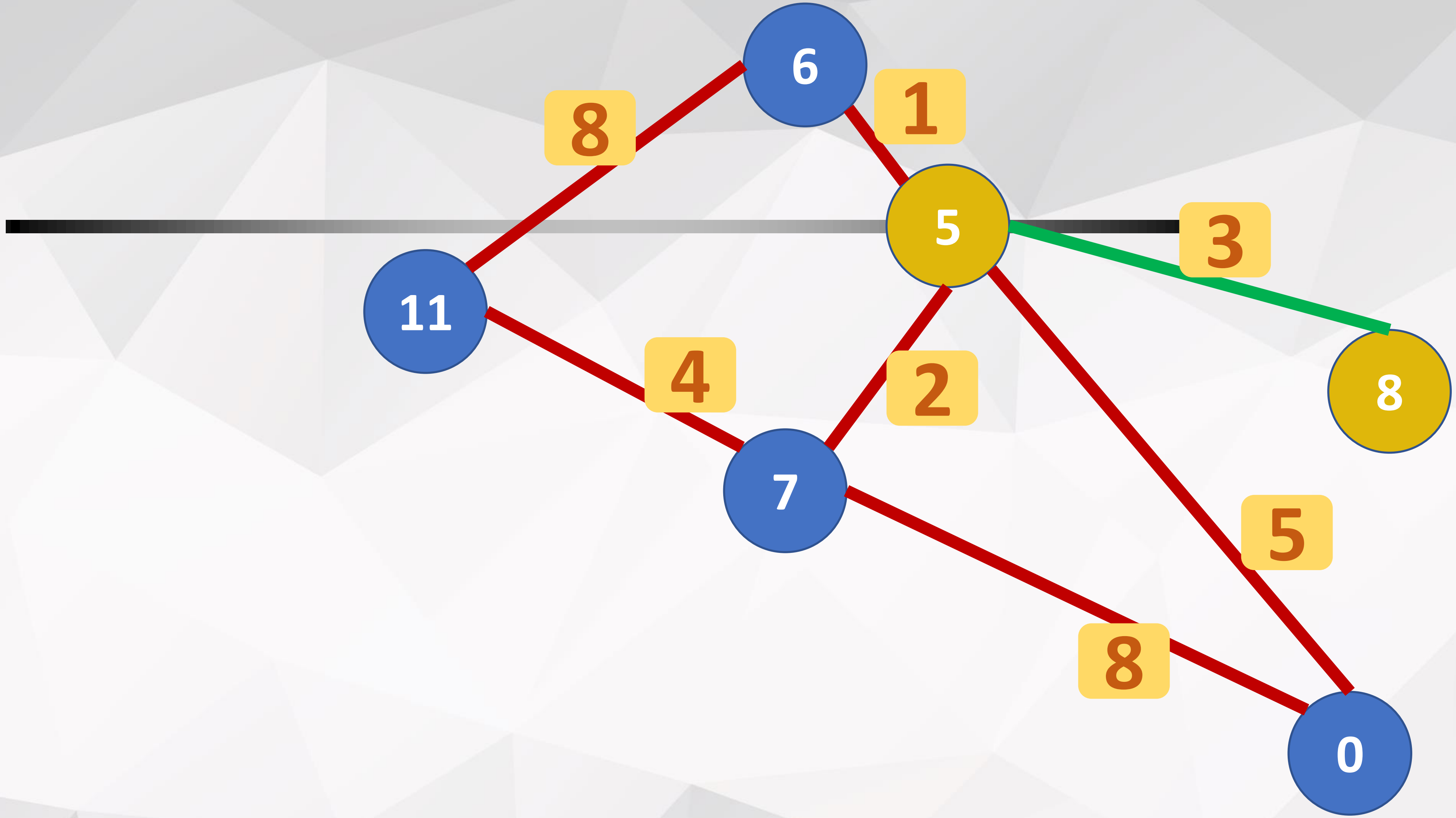
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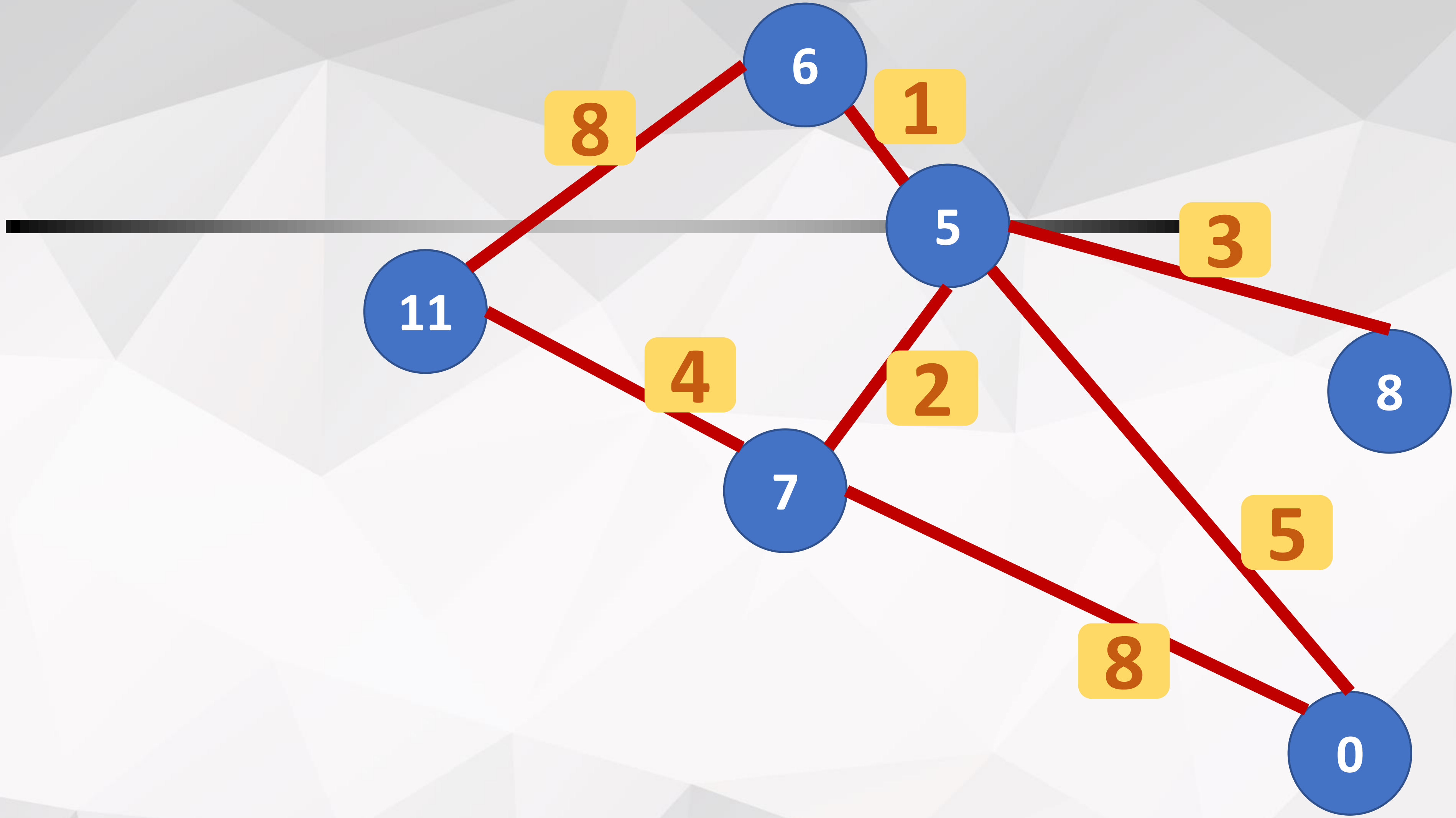
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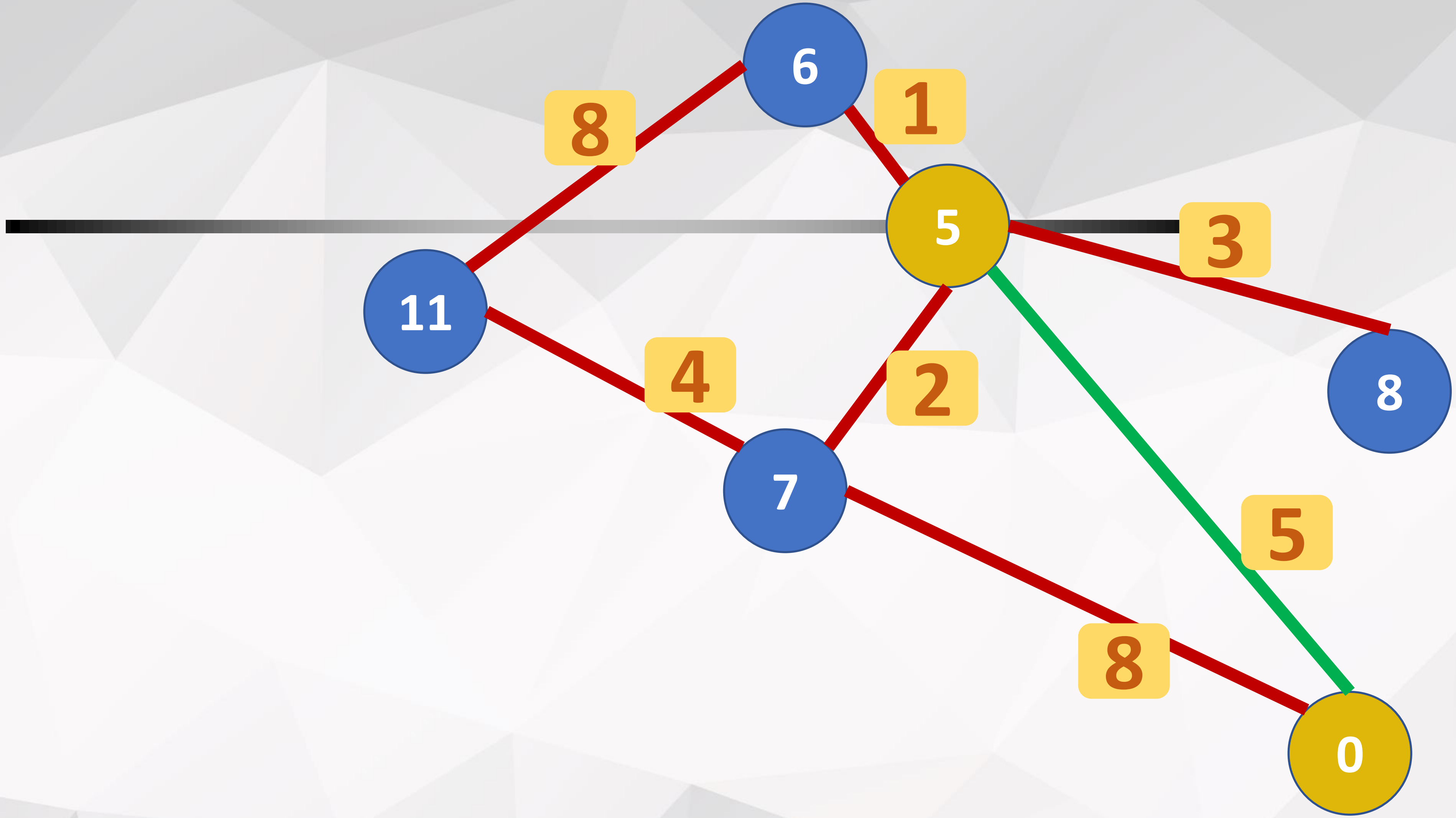
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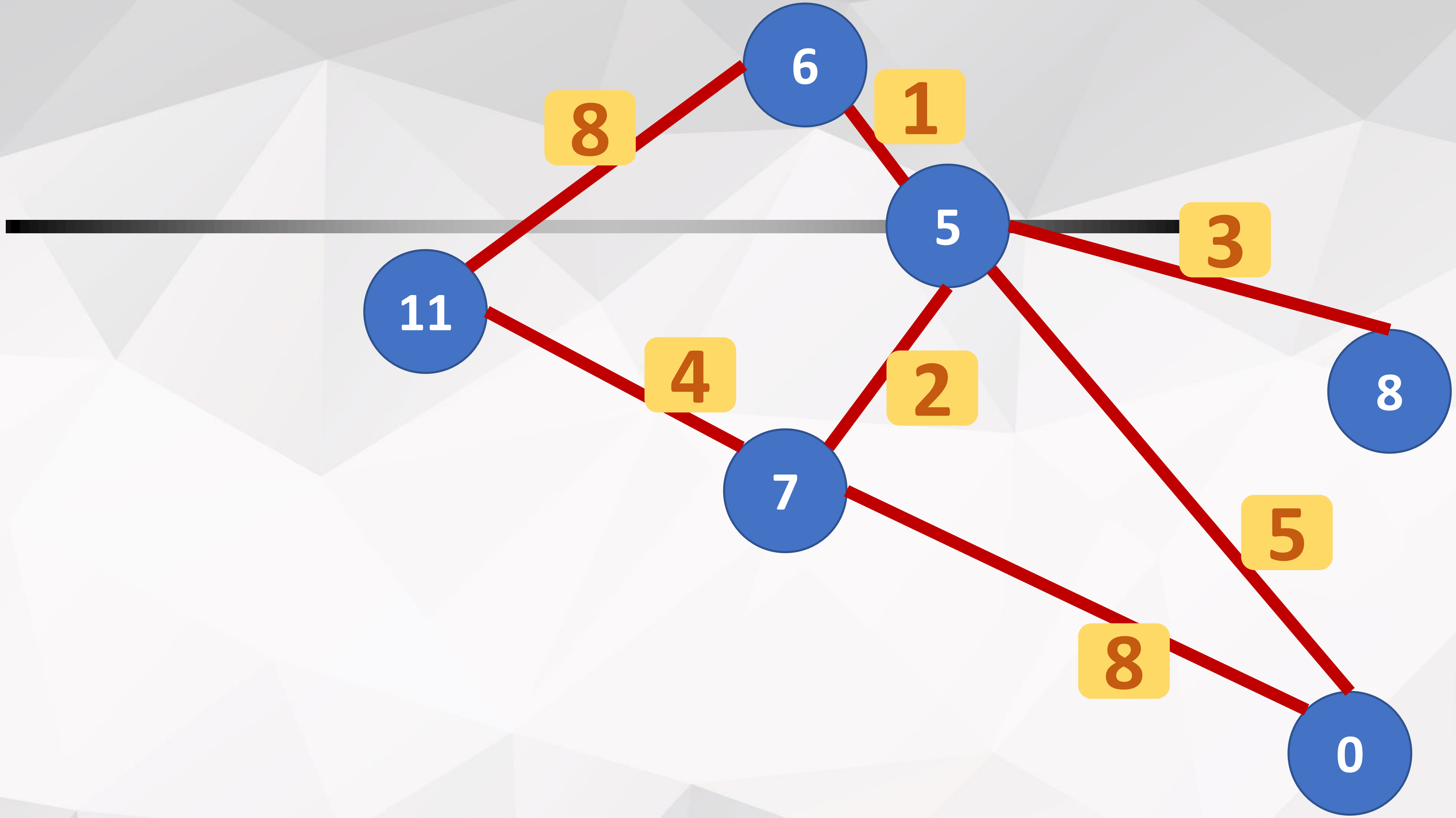
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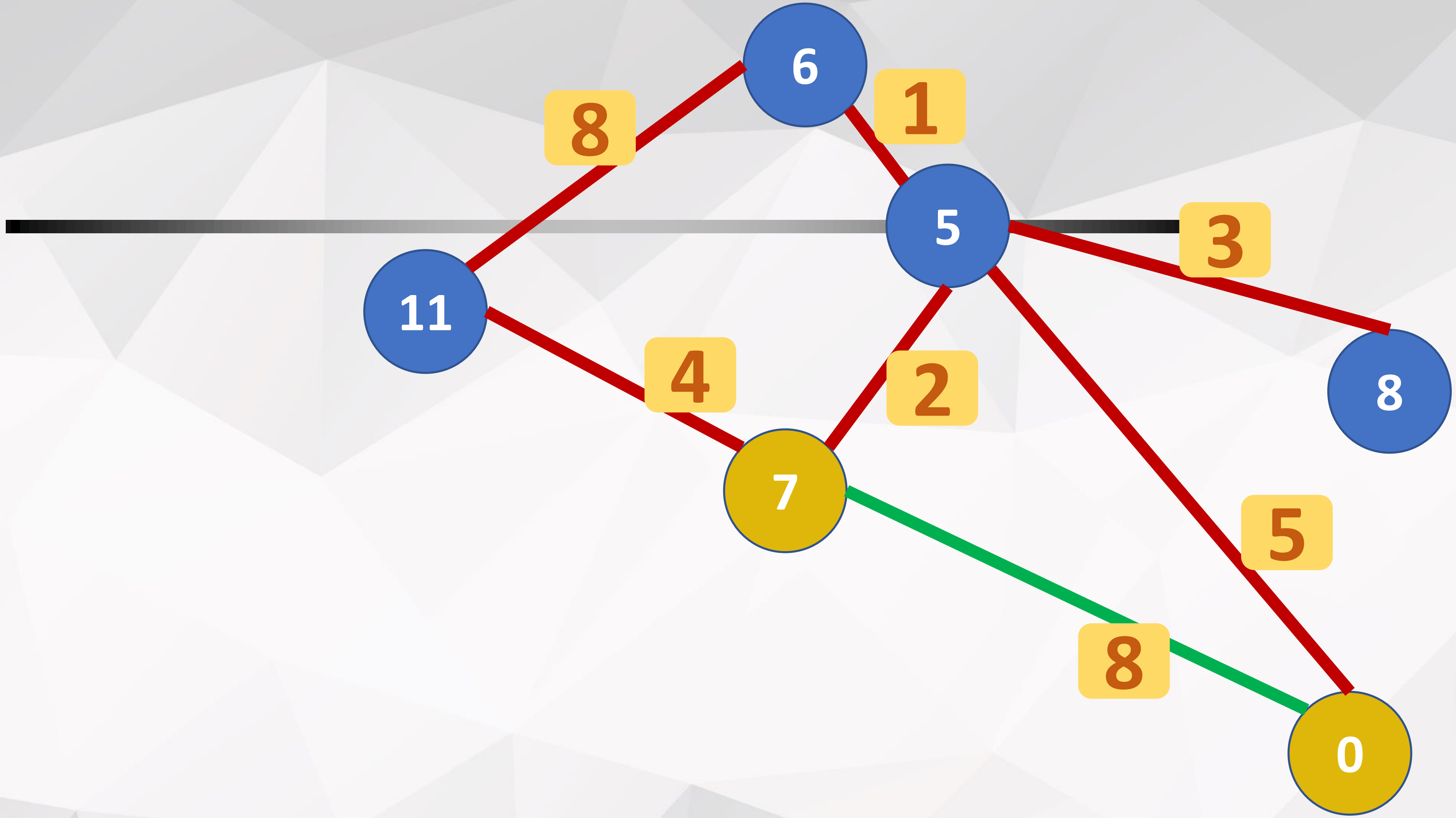
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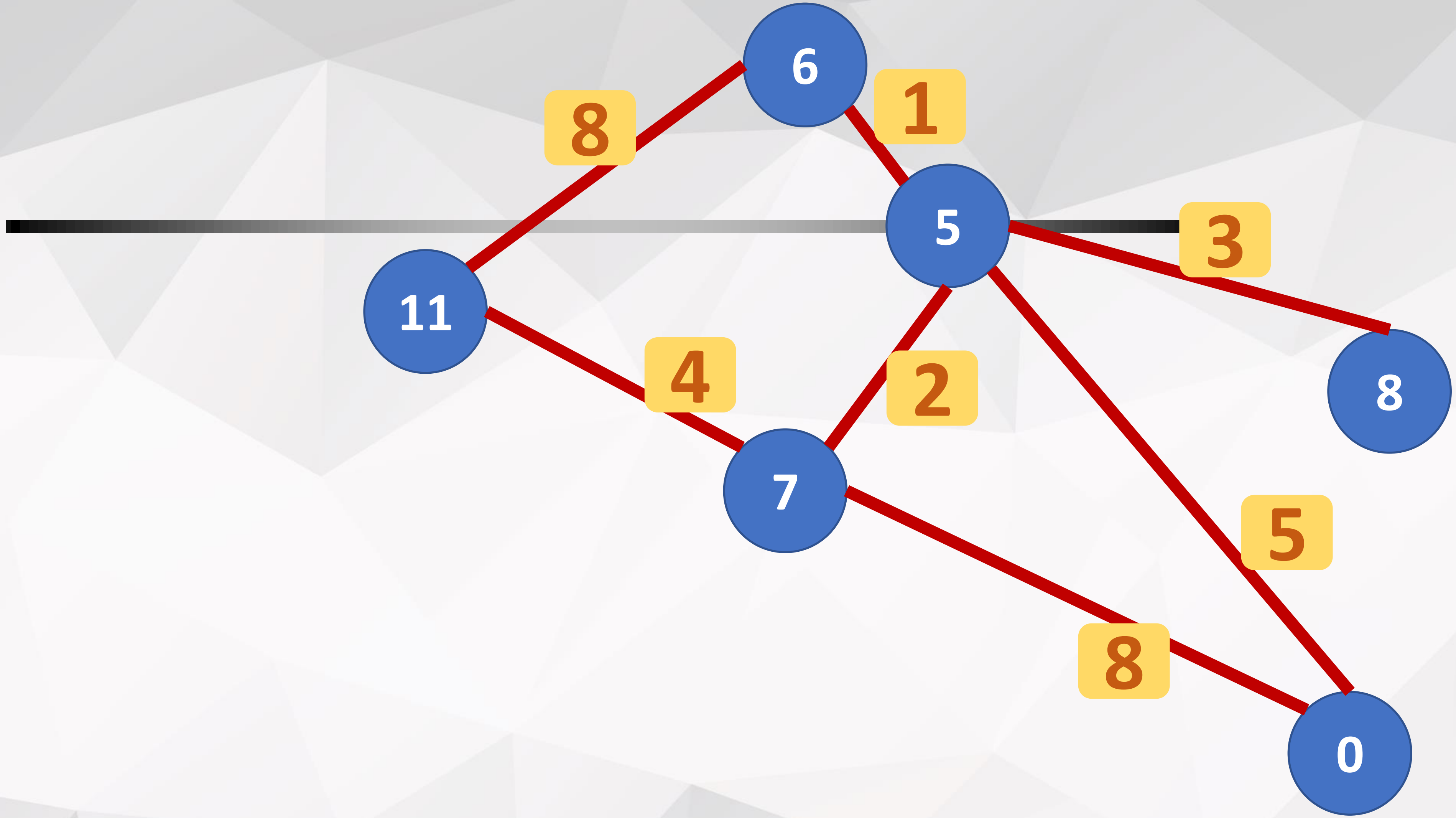
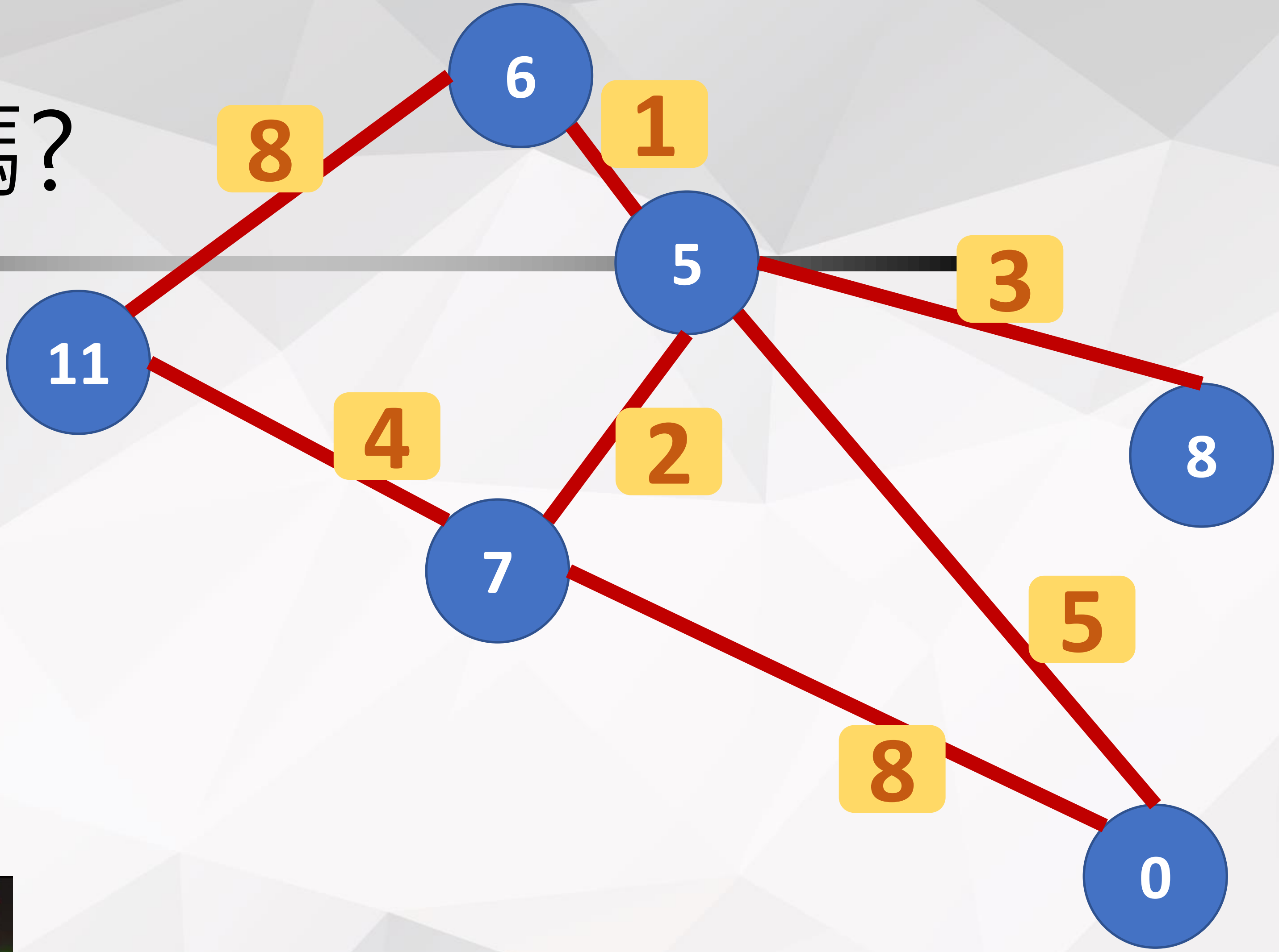

6
8
1
5
3
11
4
2
8
7
5
8
0

結束了嗎?



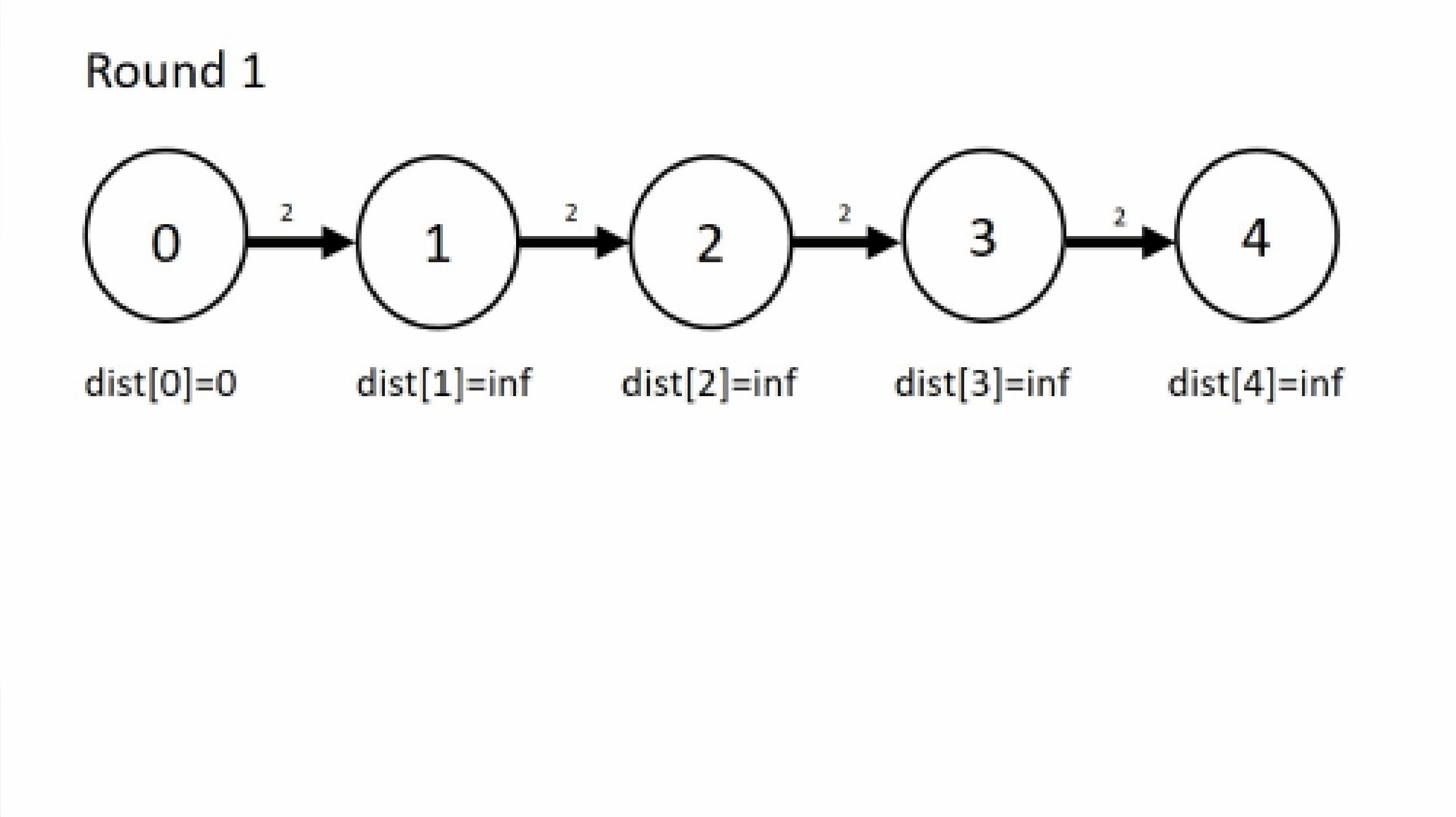
結束了嗎?

結束了。

可是

要**怎麼判斷**，每個點都得到**最短路**了?

路就是一條直直的


Round 1
0
1
2
3
4
2
2
2
2
dist[0]=0
dist[1]=inf
dist[2]=inf
dist[3]=inf
dist[4]=inf

所以

- 至多要做 |V| - 1 次的全部邊 relaxtion
  - 剛剛的例子只做了 3 遍
  - 自行證明吧

負權重邊

對每邊做至多 $|V|-1$ 次 relaxation 後，

要是某邊還能 relaxation，

負權重邊

對每邊做至多 |V|−1 次 relaxation 後，

要是某邊還能 relaxation，
就表示有**負權重邊**能使路徑成本一直降低。

Questions?

練習

UVa OJ 558 Wormholes

Outline

- 術語複習
  - Graph
  - Tree
- 最小生成樹
- A* 搜尋法則
- 單源最短路徑
- 全點對最短路徑

A* search

評估函數

下一步到底該往哪走?

評估函數

下一步到底該往哪走? (透過轉移方程找可走鄰點)

評估函數

下一步到底該往哪走?

走下去，會更好嗎?

評估函數

下一步到底該往哪走?

走下去，會更好嗎?

好或不好，就是由評估函數決定

得自行設計

評估函數

- g(n): 從**起**點到 n 點的成本
- h(n): 從 n 點到**終**點的成本

f(n) = g(n) + h(n): 評估函數

例如

當求帶權重圖的單源最短路徑

g(n) = 從起點到 n 的最小成本
h(n) = 0

這個是， Dijkstra 演算法

例如

當求二維平面圖的單點到單點最短路徑

g(n) = 0
h(n) = n 點到終點的歐幾里得距離

這個是， Best-first search 演算法
不是 Breadth-first search

Outline

- 術語複習
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All-Pairs Shortest Paths

Competitive Programming

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APSP

- 問**任意點**到**任意點**的最小成本

樸素解

- 利用剛才教的 SSSP 演算法們
- 對每個點都設定為源點 (source)

樸素解

- 利用剛才教的 SSSP 演算法們
- 對每個點都設定為源點 (source)
- 當然可以!

全點對最短路徑

- Floyd-Warshall's Algorithm
- Johnson's Algorithm

全點對最短路徑

- Floyd-Warshall's Algorithm
- Johnson's Algorithm

Floyd-Warshall's Algorithm

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Floyd-Warshall 實作

```
int s[maxn][maxn];

for (int i = 1; i <= N; i++)
  for (int j = 1; j <= N; j++)
    s[i][j] = G[i][j];

for (int k = 1; k <= N; k++)
  for (int i = 1; i <= N; i++)
    for (int j = 1; j <= N; j++)
      s[i][j] = min(s[i][j], s[i][k] + s[k][j]);
```

狀態/轉移方程

設定狀態 $s(\mathrm{i},\mathrm{j},k)$ 為 i 到 j 只以 $\{1,..,k\}$ 為中間點的最小成本

狀態/轉移方程

設定狀態 $s(\mathrm{i},\mathrm{j},k)$ 為 i 到 j 只以 $\{1,..,k\}$ 為中間點的最小成本

$$s(\mathrm{i},\mathrm{j},k)=\begin{cases} s(i,j,k-1) & \text{若無經過 } k \\ \mathrm{s}(\mathrm{i},k,k-1)+s(k,j,k-1) & \text{若有經過 } k \end{cases}$$

狀態/轉移方程

設定狀態 $s(\mathrm{i},\mathrm{j},k)$ 為 i 到 j 只以 $\{1,..,k\}$ 為中間點的最小成本

$$s(\mathrm{i},\mathrm{j},k)=\begin{cases} s(i,j,k-1) & \text{若無經過 } k \\ \mathrm{s}(\mathrm{i},k,k-1)+s(k,j,k-1) & \text{若有經過 } k \end{cases}$$

$$s(\mathrm{i},\mathrm{j},k)\ =min(s(i,j,k-1),\mathrm{s}(i,k,k-1)+s(k,j,k-1))$$

邊界

$$s(\mathrm{i},\mathrm{j},0)=\begin{cases}0 & \text{若 } i=j\\ weight(i,j) & \text{若有 } (i,j) \text{ 邊}\\ \infty & \text{若無 } (i,j) \text{ 邊}\end{cases}$$

練習

[UVa OJ 125 Numbering Paths]

全點對最短路徑

- Floyd-Warshall's Algorithm
- Johnson's Algorithm

Johnson's Algorithm

Johnson's Algorithm

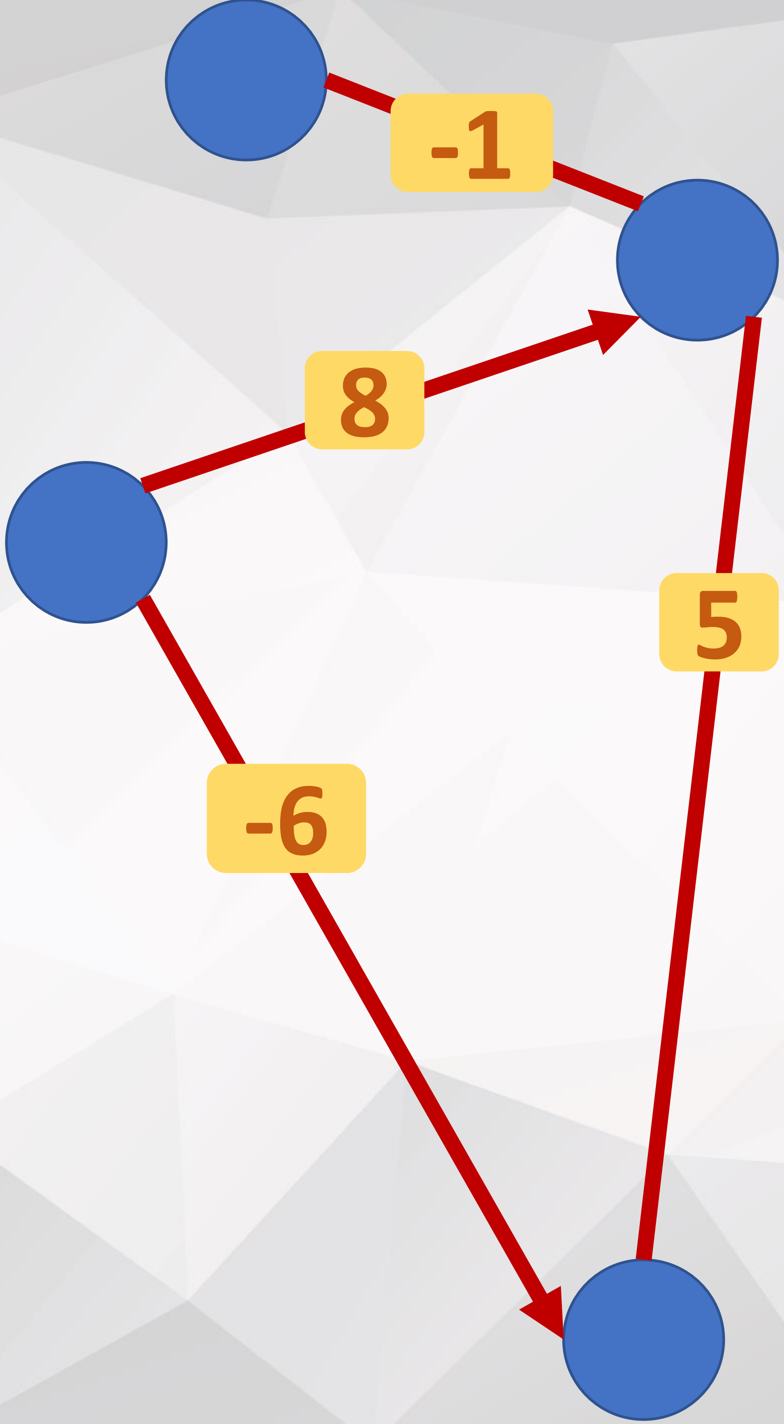
- 利用剛才教的 SSSP 演算法們
- 對每個點都設定為源點 (source)

Johnson's Algorithm

- 利用剛才教的 SSSP 演算法們
- 對每個點都設定為源點 (source)
- 接著用 Dijkstra's Algorithm 跑
  則總複雜度為 $O(|V||E| \cdot \log_2|V|)$

Johnson's Algorithm

但若圖**有負權重邊**


-1
8
5
-6

Johnson's Algorithm

但若圖**有負權重邊**
會破壞設計 Dijkstra's Algorithm 時用的**無後效性**

Johnson's Algorithm

但若圖有負權重邊
會破壞設計 Dijkstra's Algorithm 時用的無後效性

所以要想辦法把負權重邊給搞掉

Re:Weighting

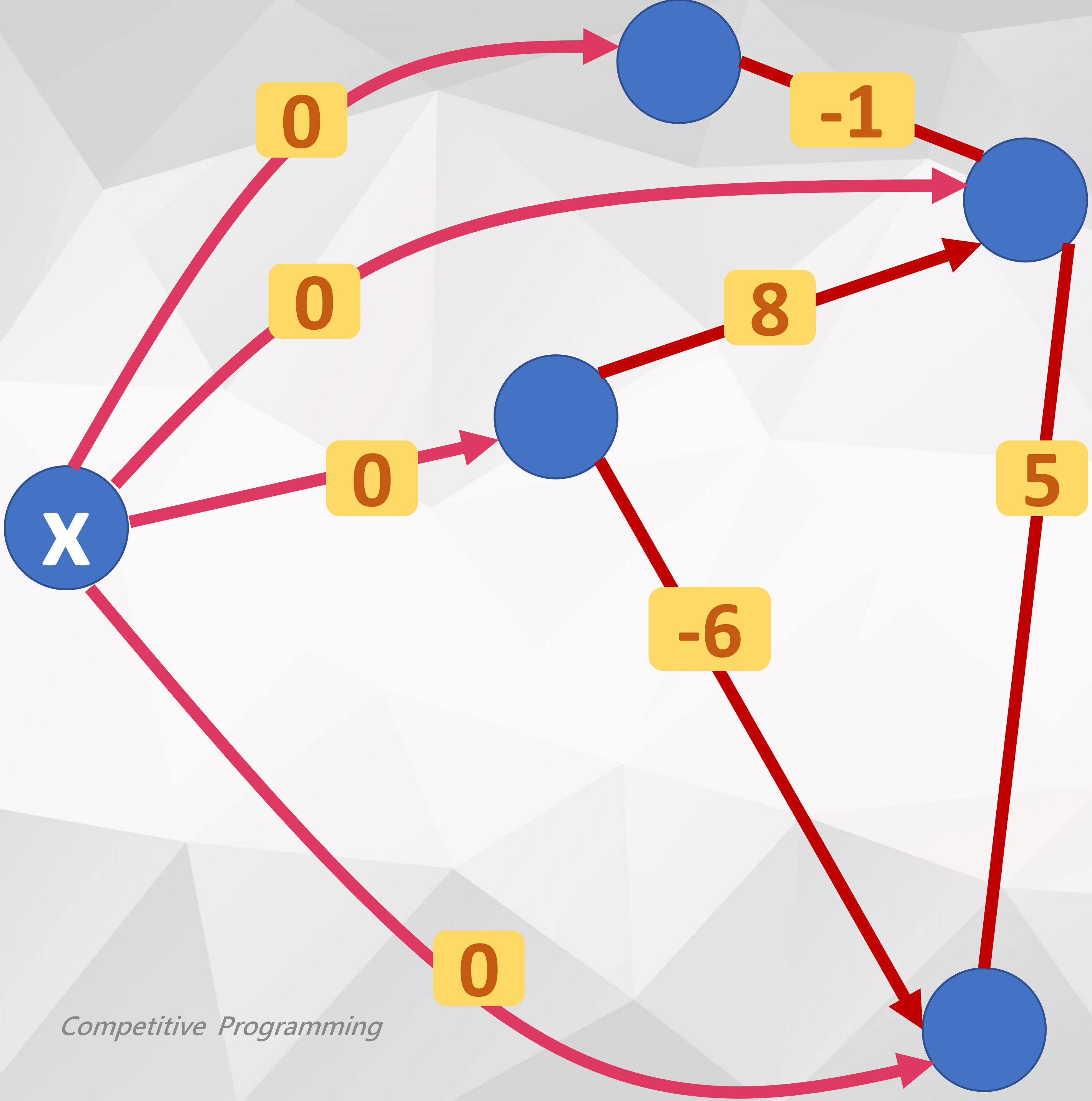
顧名思義，就是把原本的權重改成新的權重

Reweighting

顧名思義，就是把原本的權重改成新的權重

更改方式為:

- 設一個新的點 x 連到所有點，邊權重為 0

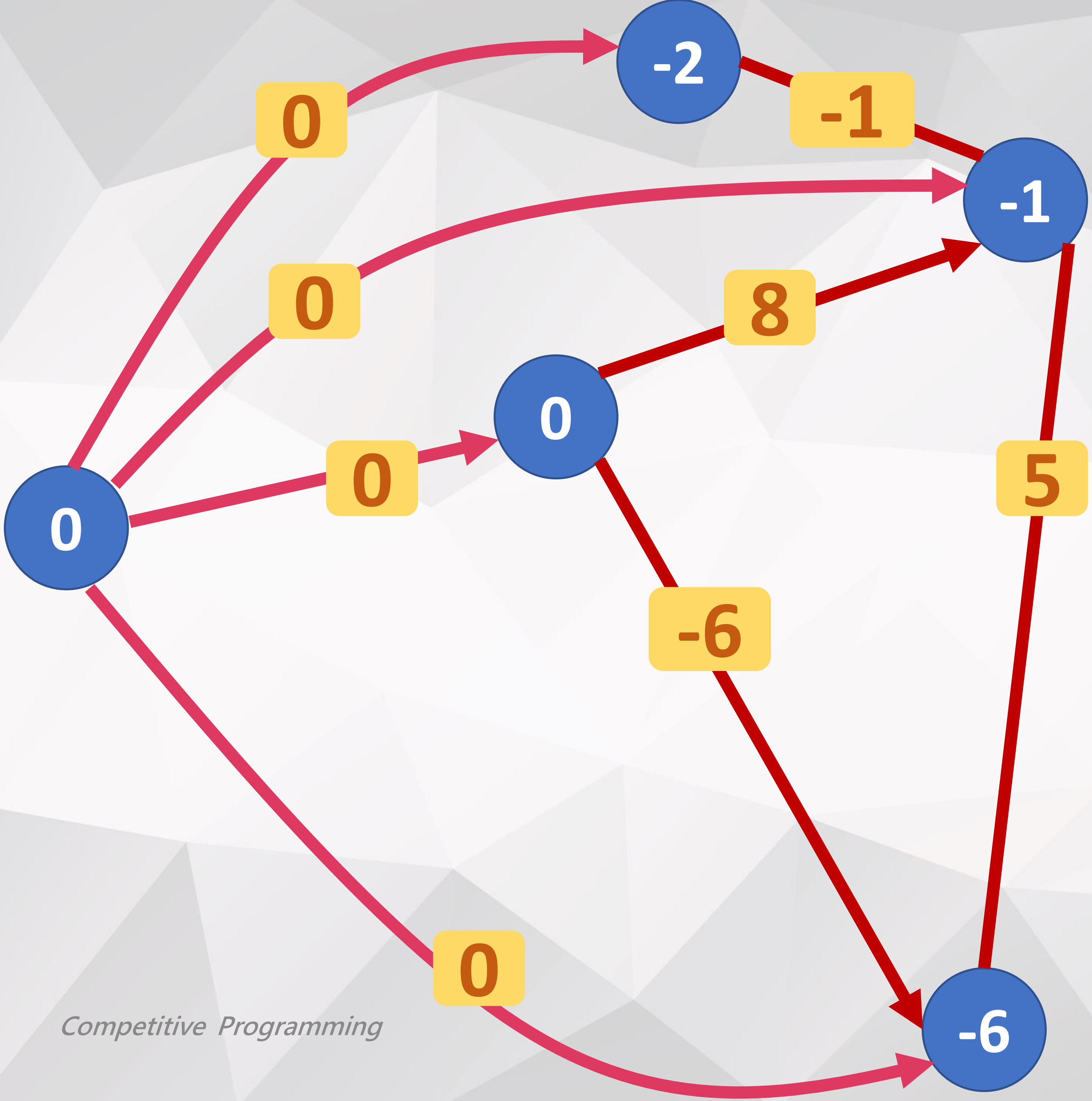

0
-1
0
8
x
0
5
-6
0

Reweighting

顧名思義，就是把原本的權重改成新的權重

更改方式為:

- 設一個新的點 x 連到所有點，邊權重為 0
- 用 Bellman-Ford's Algo 計算 x 到任意點的最短路



Competitive Programming

Made by 培訓團隊群 CCNS

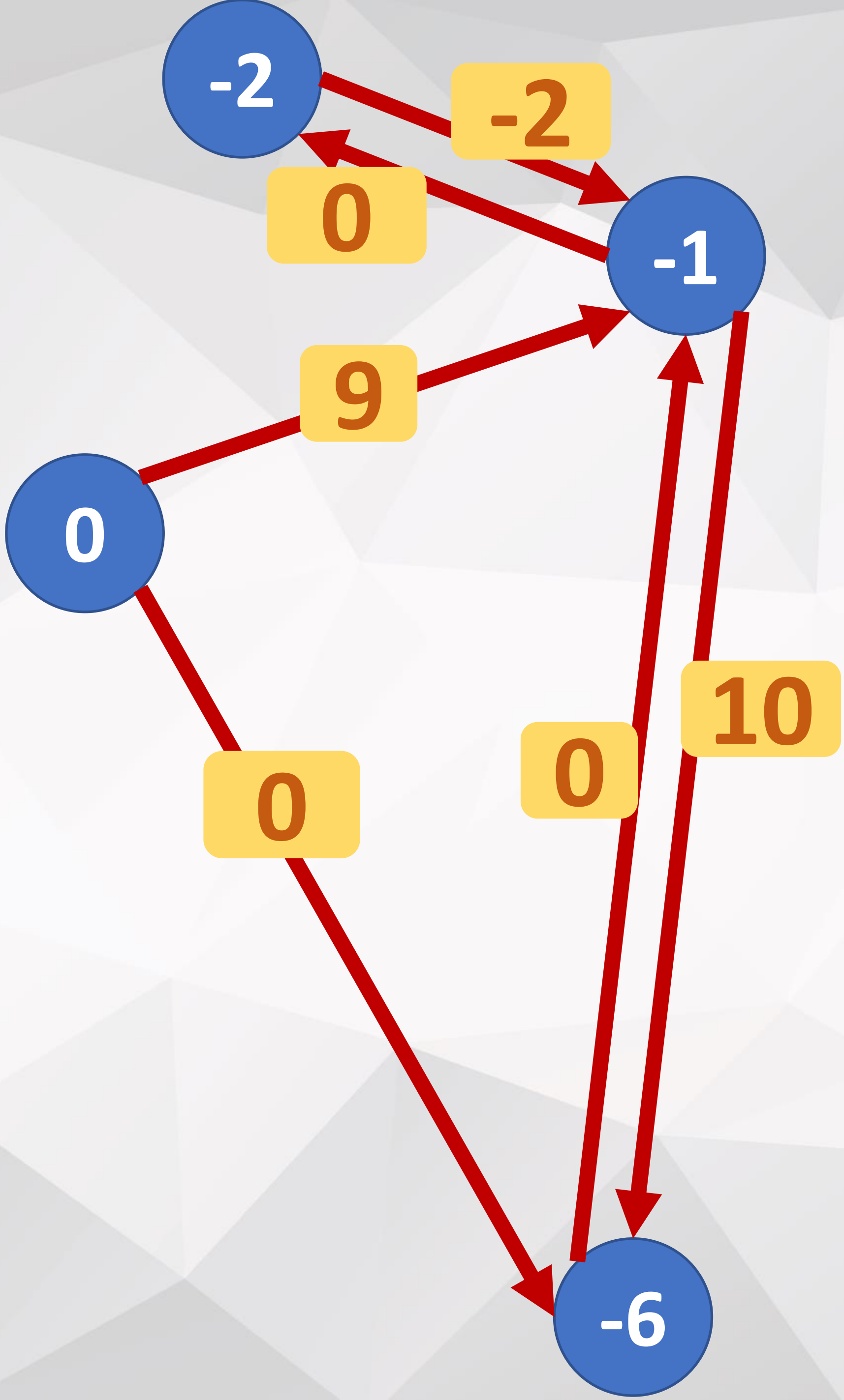
Reweighting

顧名思義，就是把原本的權重**改成**新的權重

更改方式為:

- 設一個新的點 x 連到所有點，邊權重為 0
- 用 Bellman-Ford's Algo 計算 x 到任意點的最短路
- 將所有邊權重改為 $w'(u, v) = w(u, v) + \delta(u) - \delta(v)$

$w(u, v)$ 為 u 到 v 的邊權重，$\delta(u)$ 為 x 到 u 最短路


-2
-2
0
-1
9
0
10
0
0
-6

Johnson's Algorithm

將 reweight 做完後
就可直接用 Dijkstra's Algorithm 找 APSP 的解

Johnson's Algorithm

將 reweight 做完後
就可直接用 Dijkstra's Algorithm 找 APSP 的解

為甚麼可以？
難道權重改變後原本的最短路徑不會變動嗎？

Reweight lemma

假設 u 到 v **原最短路徑**為 $v_0v_1..v_n$，且 $v_0 = u, v_n = v$
則 reweighted 的路徑權重為

$$\sum_{i=0}^{n-1} w'(v_i, v_{i+1}) = \sum_{i=0}^{n-1} w(v_i, v_{i+1}) + \delta(v_0) - \delta(v_n)$$

Reweight lemma

對於 u 到 v 的任意路徑 $t_0 t_1 .. t_n$，且 $t_0 = u, t_n = v$ reweighted 的路徑權重為

$$\sum_{i=0}^{n-1} w'(t_i, t_{i+1}) = \sum_{i=0}^{n-1} w(t_i, t_{i+1}) + \delta(t_0) - \delta(t_n)$$

Reweight lemma

根據最短路徑有

$$\sum_{i=0}^{n-1} w(v_i, v_{i+1}) \le \sum_{i=0}^{n-1} w(t_i, t_{i+1})$$

能推得

$$\sum_{i=0}^{n-1} w(v_i, v_{i+1}) + \delta(u) - \delta(v) \le \sum_{i=0}^{n-1} w(t_i, t_{i+1}) + \delta(u) - \delta(v)$$

Reweight lemma

根據最短路徑有

$$\sum_{i=0}^{n-1} w(v_i, v_{i+1}) \leq \sum_{i=0}^{n-1} w(t_i, t_{i+1})$$

能推得

$$\sum_{i=0}^{n-1} w(v_i, v_{i+1}) + \delta(u) - \delta(v) \leq \sum_{i=0}^{n-1} w(t_i, t_{i+1}) + \delta(u) - \delta(v)$$

Reweight lemma

根據最短路徑有

$$\sum_{i=0}^{n-1} w(v_i, v_{i+1}) \le \sum_{i=0}^{n-1} w(t_i, t_{i+1})$$

能推得

$$\sum_{i=0}^{n-1} w(v_i, v_{i+1}) + \delta(v_0) - \delta(v_n) \le \sum_{i=0}^{n-1} w(t_i, t_{i+1}) + \delta(t_0) - \delta(t_n)$$

Reweight lemma

故 reweighted 圖的任意點對最短路徑
與
原圖的任意點對最短路徑
是相同的

Johnson's Algorithm

將 reweight 做完後
就可直接用 Dijkstra's Algorithm 找 APSP 的解

接著將所有點對 (u, v) 最短路**權重減去 (δ(u) – δ(v))**
就得到原問題的解了。

Johnson's Algorithm

結合了 Bellman-ford 與 Dijkstra 的演算法

所以複雜度為兩個演算法複雜度相加

Questions?